Detection & Estimation Theory: Lectures 1 and 2

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Outline

- Problem Definition & Basic Assumptions
- Formulation of (Bayesian) Risk
- Minimization of Bayesian Risk: Likelihood Ratio Test (LRT)
- LRT Special Cases: Maximum Aposteriori Probability (MAP) Rule
  - Maximum Likelihood (ML)
- Example

Binary Hypothesis Testing: Problem Definition

- Given a collection of measurements $\mathbf{y} \in \mathcal{Y}$, find *optimal* decision function $\delta(\mathbf{y})$ that splits observation space $\mathcal{Y}$ in two disjoint regions $\mathcal{Y}_0,\ \mathcal{Y}_1$:

$$\delta(\mathbf{y}) = \begin{cases} 0, & \text{under hypothesis } H_0 \text{ (i.e., } \mathbf{y} \in \mathcal{Y}_0) \\ 1, & \text{under hypothesis } H_1 \text{ (i.e., } \mathbf{y} \in \mathcal{Y}_1) \end{cases} \tag{1}$$

  where hypothesis 0 and hypothesis 1 are denoted by H_0, H_1, respectively.

- Vector $\mathbf{y}$ denotes a collection of measurements.
- **Continuous case**: $f_{\mathbf{y}|H_j}\left(\mathbf{y}|H_j\right),\ j \in \{0, 1\}$, i.e., conditional probability density function (pdf) is known and $f_{\mathbf{y}|H_0}\left(\mathbf{y}|H_0\right) \neq f_{\mathbf{y}|H_1}\left(\mathbf{y}|H_1\right)$.
- **Discrete case**: $\Pr\left(\mathbf{y}|H_j\right),\ j \in \{0, 1\}$, i.e., conditional probability mass function (pmf) is known and $\Pr\left(\mathbf{y}|H_0\right) \neq \Pr\left(\mathbf{y}|H_1\right)$.
- Priors $\pi_0 \triangleq \Pr\left(H_0\right) = 1 - \pi_1$, $\pi_1 \triangleq \Pr\left(H_1\right)$ are known.

Binary Hypothesis Testing: Problem Definition

- Given a collection of measurements $\mathbf{y} \in \mathcal{Y}$, find *optimal* decision function $\delta(\mathbf{y})$ that splits observation space $\mathcal{Y}$ in two disjoint regions $\mathcal{Y}_0,\ \mathcal{Y}_1$:

$$\delta(\mathbf{y}) = \begin{cases} 0, & \text{under hypothesis } H_0 \text{ (i.e., } \mathbf{y} \in \mathcal{Y}_0) \\ 1, & \text{under hypothesis } H_1 \text{ (i.e., } \mathbf{y} \in \mathcal{Y}_1) \end{cases} \tag{2}$$

  where hypothesis 0 and hypothesis 1 are denoted by H_0, H_1, respectively.

- We need an *optimality* criterion!

- Bayes comes to help: all uncertainties are quantifiable, all costs and benefits of decision can be measured!

Formulation of (Bayesian) Risk

- Define cost C_{ij} of deciding that H_i holds, when hypothesis H_j is true, $i, j \in \{0, 1\}$.
- Define

$$\Pr\left(\delta(\mathbf{y}) = i | H_j\right) =$$

$$\triangleq \Pr\left(\mathbf{y} \in \mathcal{Y}_i | H_j\right) = \begin{cases} \int_{\mathbf{y}\in\mathcal{Y}_i} f_{\mathbf{y}|H_j}\left(\mathbf{y}|H_j\right) d\mathbf{y}, & \text{(continuous case)} \\ \sum_{\mathbf{y}\in\mathcal{Y}_i} \Pr\left(\mathbf{y}|H_j\right). & \text{(discrete case)} \end{cases} \quad (3)$$

- We are ready to define the *conditional* Bayesian Risk $R\left(\cdot|\cdot\right)$ for decision rule $\delta(\mathbf{y})$ under hypothesis H_j:

$$\begin{aligned} R(\delta(\mathbf{y})|\mathrm{H}_j) &= C_{1j} \Pr(\delta(\mathbf{y}) = 1|H_j) + C_{0j} \Pr(\delta(\mathbf{y}) = 0|H_j) \\ &= \sum_{i=0}^{1} C_{ij} \Pr(\delta(\mathbf{y}) = i|H_j). \end{aligned} \quad (4)$$

Formulation of (Bayesian) Risk

- *Conditional* Bayesian Risk $R(\cdot|\cdot)$ for decision rule $\delta(\mathbf{y})$ under hypothesis H_j:

$$R(\delta(\mathbf{y})|\mathrm{H}_j) = \sum_{i=0}^{1} C_{ij} \, \Pr\left(\delta(\mathbf{y}) = i|H_j\right). \tag{5}$$

- Thus, the average unconditional Bayesian cost of decision rule $\delta(\mathbf{y})$ follows:

$$\begin{aligned}
R(\delta(\mathbf{y})) &= R(\delta(\mathbf{y})|H_0) \ \Pr(H_0) + R(\delta(\mathbf{y})|H_1) \ \Pr(H_1) && (6)\\
&= R(\delta(\mathbf{y})|H_0) \ \pi_0 + R(\delta(\mathbf{y})|H_1) \ \pi_1 && (7)\\
&= \sum_{j=0}^{1} R(\delta(\mathbf{y})|H_j) \ \pi_j && (8)\\
&\overset{Eq. (5)}{=} \sum_{i=0}^{1}\sum_{j=0}^{1} \pi_j \ C_{ij} \ \Pr\left(\delta(\mathbf{y}) = i|H_j\right) && (9)
\end{aligned}$$

Formulation of (Bayesian) Risk

- Exploiting the fact that $\mathcal{Y}_0 \cup \mathcal{Y}_1 = \mathcal{Y}$ and $\mathcal{Y}_0 \cap \mathcal{Y}_1 = \emptyset$:

$$\Pr\left(\delta(\mathbf{y}) = 0 | H_j\right) + \Pr\left(\delta(\mathbf{y}) = 1 | H_j\right) = 1 \Leftrightarrow \tag{10}$$

$$\Pr\left(\mathbf{y} \in \mathcal{Y}_0 | H_j\right) = 1 - \Pr\left(\mathbf{y} \in \mathcal{Y}_1 | H_j\right) \tag{11}$$

- Average Bayesian cost of decision rule $\delta(\mathbf{y})$:

$$R(\delta(\mathbf{y})) = \sum_{i=0}^{1} \sum_{j=0}^{1} \pi_j \; C_{ij} \; \Pr\left(\delta(\mathbf{y}) = i | H_j\right) \tag{12}$$

$$= \sum_{j=0}^{1} \sum_{i=0}^{1} \pi_j \; C_{ij} \; \Pr\left(\mathbf{y} \in \mathcal{Y}_i | H_j\right) \tag{13}$$

$$= \sum_{j=0}^{1} \pi_j \; C_{0j} \; \Pr\left(\mathbf{y} \in \mathcal{Y}_0 | H_0\right) + \pi_j \; C_{1j} \; \Pr\left(\mathbf{y} \in \mathcal{Y}_1 | H_j\right) \tag{14}$$

$$\stackrel{(11)}{=} \sum_{j=0}^{1} \pi_j \; C_{0j} + \sum_{j=0}^{1} \pi_j \; \left(C_{1j} - C_{0j}\right) \; \Pr\left(\mathbf{y} \in \mathcal{Y}_1 | H_j\right) \tag{15}$$

Minimization of Bayesian Risk

- Average Bayesian cost (Bayesian Risk) of decision rule $\delta(\mathbf{y})$:

$$R(\delta(\mathbf{y})) = \sum_{j=0}^{1} \pi_j \; C_{0j} + \sum_{j=0}^{1} \pi_j \; \left(C_{1j} - C_{0j}\right) \; \Pr\left(\mathbf{y} \in \mathcal{Y}_1 | H_j\right) \quad (16)$$

- Notice that the first sum is independent of the measurement data $\mathbf{y}$. The *optimal* decision rule should perform the following minimization:

$$\min_{\delta(\mathbf{y})} \; R(\delta(\mathbf{y})) \quad (17)$$

- Two cases: $\mathbf{y}$ continuous or discrete (solution will be found the same!).

Minimization of Bayesian Risk

- Continuous case: $\Pr\left(\mathbf{y} \in \mathcal{Y}_1 | H_j\right) = \int_{\mathbf{y} \in \mathcal{Y}_1} f_{\mathbf{y}|H_j}\left(\mathbf{y}|H_j\right) d\mathbf{y}$,
- Bayesian Risk of decision rule $\delta(\mathbf{y})$:

$$R(\delta(\mathbf{y})) = \sum_{j=0}^{1} \pi_j \; C_{0j} + \sum_{j=0}^{1} \pi_j \; \left(C_{1j} - C_{0j}\right) \; \int_{\mathbf{y} \in \mathcal{Y}_1} f_{\mathbf{y}|H_j}\left(\mathbf{y}|H_j\right) d\mathbf{y}$$

$$= \sum_{j=0}^{1} \pi_j \; C_{0j} + \int_{\mathbf{y} \in \mathcal{Y}_1} \; \sum_{j=0}^{1} \pi_j \; \left(C_{1j} - C_{0j}\right) f_{\mathbf{y}|H_j}\left(\mathbf{y}|H_j\right) d\mathbf{y} \quad (18)$$

- Remember that $\delta(\mathbf{y})$ controls what data $\mathbf{y}$ is allocated to $\mathcal{Y}_1$ and what data to $\mathcal{Y}_0$. From Eq. (18), $R(\delta(\mathbf{y}))$ is minimized when then integrand of (18) is minimized (i.e., negative or zero):

$$\delta_B(\mathbf{y}) = \arg\min_{\delta(\mathbf{y})} \; R(\delta(\mathbf{y})) \Leftrightarrow$$

$$\text{select } \mathcal{Y}_1 : \left\{ \mathbf{y} \in \mathcal{Y} : \sum_{j=0}^{1} \pi_j \; \left(C_{1j} - C_{0j}\right) f_{\mathbf{y}|H_j}\left(\mathbf{y}|H_j\right) \leq 0 \right\} \quad (19)$$

Minimization of Bayesian Risk

- Discete case: $\Pr(\mathbf{y} \in \mathcal{Y}_1 | H_j) = \sum_{\mathbf{y} \in \mathcal{Y}_1} \Pr(\mathbf{y} | H_j)$,
- Bayesian Risk of decision rule $\delta(\mathbf{y})$:

$$R(\delta(\mathbf{y})) = \sum_{j=0}^{1} \pi_j \; C_{0j} + \sum_{j=0}^{1} \pi_j \;\; (C_{1j} - C_{0j}) \;\; \sum_{\mathbf{y} \in \mathcal{Y}_1} \Pr(\mathbf{y} | H_j)$$

$$= \sum_{j=0}^{1} \pi_j \; C_{0j} + \sum_{\mathbf{y} \in \mathcal{Y}_1} \;\; \sum_{j=0}^{1} \pi_j \;\; (C_{1j} - C_{0j}) \Pr(\mathbf{y} | H_j) \, d\mathbf{y} \tag{20}$$

- Similarly to the continuous case, $R(\delta(\mathbf{y}))$ is minimized when then integrand of (20) is minimized (i.e., negative or zero):

$$\delta_B(\mathbf{y}) = \arg\min_{\delta(\mathbf{y})} \; R(\delta(\mathbf{y})) \Leftrightarrow$$

$$\text{select } \mathcal{Y}_1 : \left\{ \mathbf{y} \in \mathcal{Y} : \sum_{j=0}^{1} \pi_j \;\; (C_{1j} - C_{0j}) \Pr(\mathbf{y} | H_j) \leq 0 \right\} \tag{21}$$

Minimization of Bayesian Risk

- Define likelihood ratio (LR) for continuous or discrete case:

$$L(\mathbf{y}) = \frac{f_{\mathbf{y}|H_1}(\mathbf{y}|H_1)}{f_{\mathbf{y}|H_0}(\mathbf{y}|H_0)} \text{(continuous case)}, L(\mathbf{y}) = \frac{\Pr(\mathbf{y}|H_1)}{\Pr(\mathbf{y}|H_0)} \text{(discrete case)}.$$

- We can safely assume that the LR is finite positive for all cases of interest (see below):
  - LR numerator and denominator both positive: LRT finite positive.
  - LR numerator and denominator both zero: this is impossible, since in that case $\mathbf{y} \notin \mathcal{Y}$ (and we have also assumed that $\Pr(\mathbf{y}|H_1) \neq \Pr(\mathbf{y}|H_0)$).
  - either numerator or denominator (only one of the two) is zero; if numerator is zero then that particular $\mathbf{y}$ cannot occur under H_1; similarly, if denominator is zero then that $\mathbf{y}$ cannot occur under H_0.

Minimization of Bayesian Risk: Likelihood Ratio Test

- We further assume that $C_{01} > C_{11}$, i.e., the cost of wrong decision is strictly higher than the cost of correct decision.
- Continuous case:

$$\text{select } \mathcal{Y}_1 : \left\{ \mathbf{y} \in \mathcal{Y} : \sum_{j=0}^{1} \pi_j \; (C_{1j} - C_{0j}) \, f_{\mathbf{y}|H_j}(\mathbf{y}|H_j) \le 0 \right\} \Leftrightarrow$$

$$\Leftrightarrow \sum_{j=0}^{1} \pi_j \; (C_{1j} - C_{0j}) \, f_{\mathbf{y}|H_j}(\mathbf{y}|H_j) \overset{H_1}{\le} 0 \tag{22}$$

$$\Leftrightarrow \pi_0 \; (C_{10} - C_{00}) \, f_{\mathbf{y}|H_0}(\mathbf{y}|H_0) \overset{H_1}{\le} -\pi_1 \; (C_{11} - C_{01}) \, f_{\mathbf{y}|H_1}(\mathbf{y}|H_1)$$

$$\overset{(C_{01}-C_{11})>0}{\Leftrightarrow} \frac{f_{\mathbf{y}|H_1}(\mathbf{y}|H_1)}{f_{\mathbf{y}|H_0}(\mathbf{y}|H_0)} \overset{H_1}{\ge} \frac{C_{10} - C_{00}}{C_{01} - C_{11}} \; \frac{\pi_0}{\pi_1} \triangleq \tau \tag{23}$$

$$\Leftrightarrow L(\mathbf{y}) \overset{H_1}{\ge} \tau \tag{24}$$

Minimization of Bayesian Risk: Likelihood Ratio Test

- Continuous case:

$$L(\mathbf{y}) \triangleq \frac{f_{\mathbf{y}|H_1}(\mathbf{y}|H_1)}{f_{\mathbf{y}|H_0}(\mathbf{y}|H_0)} \overset{H_1}{\geq} \frac{C_{10}-C_{00}}{C_{01}-C_{11}} \frac{\pi_0}{\pi_1} \triangleq \tau \tag{25}$$

$$\Leftrightarrow L(\mathbf{y}) \overset{H_1}{\geq} \tau \tag{26}$$

  Notice that the values of $\mathbf{y}$ where the integrand goes to zero (or equivalently the (LR) ratio is equal to τ) do not matter; some can be allocated to $\mathcal{Y}_1$ and some (or none) to $\mathcal{Y}_0$).

- Discrete case - with similar reasoning, minimization in Eq. (21) offers the following LR test:

$$L(\mathbf{y}) \triangleq \frac{\Pr(\mathbf{y}|H_1)}{\Pr(\mathbf{y}|H_0)} \overset{H_1}{\geq} \frac{C_{10}-C_{00}}{C_{01}-C_{11}} \frac{\pi_0}{\pi_1} \triangleq \tau \tag{27}$$

$$\Leftrightarrow L(\mathbf{y}) \overset{H_1}{\geq} \tau \tag{28}$$

Minimization of Bayesian Risk: Likelihood Ratio Test

- Thus, optimum Bayesian decision rule $\delta_B(\mathbf{y})$, i.e., rule that minimizes Bayes risk, can be written as follows:

$$\delta_B(\mathbf{y}) = \begin{cases} 1, & \text{if } L(\mathbf{y}) \geq \tau, \\ 0, & \text{if } L(\mathbf{y}) < \tau, \end{cases} \tag{29}$$

or more compactly,

$$L(\mathbf{y}) \overset{H_1}{\geq} \tau. \tag{30}$$

Likelihood Ratio Test (LRT) & Symmetric Costs

- Set symmetric costs, i.e., 1 for (any) erroneous detection and 0 for (any) correct decision:

$$C_{ij} = 1 - \delta_{ij} = \begin{cases} 0, & i = j, \\ 1, & i \neq j, \end{cases} \tag{31}$$

where δ_{ij} denotes the Kronecker delta. For such costs, Bayesian Risk is equivalent to probability of error! From Eq. (4):

$$\begin{aligned}
R(\delta(\mathbf{y})|\mathrm{H}_j) &= C_{1j} \Pr(\delta(\mathbf{y}) = 1|H_j) + C_{0j} \Pr(\delta(\mathbf{y}) = 0|H_j) \Rightarrow \\
R(\delta(\mathbf{y})|\mathrm{H}_0) &= C_{10} \Pr(\delta(\mathbf{y}) = 1|H_0) + C_{00} \Pr(\delta(\mathbf{y}) = 0|H_0) \\
&= \Pr(\delta(\mathbf{y}) = 1|H_0) \equiv \Pr(\mathrm{error}|H_0). && (32) \\
R(\delta(\mathbf{y})|\mathrm{H}_1) &= C_{11} \Pr(\delta(\mathbf{y}) = 1|H_1) + C_{01} \Pr(\delta(\mathbf{y}) = 0|H_1) \\
&= \Pr(\delta(\mathbf{y}) = 0|H_1) \equiv \Pr(\mathrm{error}|H_1) \Rightarrow && (33) \\
R(\delta(\mathbf{y})) &= R(\delta(\mathbf{y})|\mathrm{H}_0)\ \pi_0 + R(\delta(\mathbf{y})|\mathrm{H}_1)\ \pi_1 && (34) \\
&= \Pr(\mathrm{error}|H_0)\ \pi_0 + \Pr(\mathrm{error}|H_1)\ \pi_1 \equiv \Pr(\mathrm{error}).
\end{aligned}$$

- Set symmetric costs $C_{ij} = 1 - \delta_{ij}$, as before. For such costs, Bayesian Risk is equivalent to probability of error!
- Continuous case:[1]

$$L(\mathbf{y}) \triangleq \frac{f_{\mathbf{y}|H_1}(\mathbf{y}|H_1)}{f_{\mathbf{y}|H_0}(\mathbf{y}|H_0)} \overset{H_1}{\geq} \tau \triangleq \frac{C_{10} - C_{00}}{C_{01} - C_{11}} \frac{\pi_0}{\pi_1} = \frac{(1-0)}{(1-0)} \frac{\pi_0}{\pi_1} = \frac{\pi_0}{\pi_1}$$

$$\Leftrightarrow f_{\mathbf{y}|H_1}(\mathbf{y}|H_1)\,\pi_1 \overset{H_1}{\geq} f_{\mathbf{y}|H_0}(\mathbf{y}|H_0)\,\pi_0 \tag{35}$$

$$\Leftrightarrow \frac{f_{\mathbf{y}|H_1}(\mathbf{y}|H_1)\,\pi_1}{f_{\mathbf{y}}(\mathbf{y})} \overset{H_1}{\geq} \frac{f_{\mathbf{y}|H_0}(\mathbf{y}|H_0)\,\pi_0}{f_{\mathbf{y}}(\mathbf{y})} \tag{36}$$

$$\overset{\text{Bayes}(*)}{\Leftrightarrow} \Pr(H_1|\mathbf{y}) \overset{H_1}{\geq} \Pr(H_0|\mathbf{y}) \quad \text{(MAP Rule)} \tag{37}$$

- Discrete case: same rule as above!

[1] (*) holds because the Bayes property holds for continuous distributions as well.

- Set symmetric costs $C_{ij} = 1 - \delta_{ij}$, as before and equal priors $\pi_0 = \pi_1$ (special MAP case):
- Continuous case:

$$\frac{f_{\mathbf{y}|H_1}(\mathbf{y}|H_1)\,\pi_1}{f_{\mathbf{y}}(\mathbf{y})} \overset{H_1}{\geq} \frac{f_{\mathbf{y}|H_0}(\mathbf{y}|H_0)\,\pi_0}{f_{\mathbf{y}}(\mathbf{y})} \tag{38}$$

$$\Rightarrow f_{\mathbf{y}|H_1}(\mathbf{y}|H_1) \overset{H_1}{\geq} f_{\mathbf{y}|H_0}(\mathbf{y}|H_0) \quad \text{(ML Rule)} \tag{39}$$

- Discrete case - same derivation as above:

$$\Pr(\mathbf{y}|H_1) \overset{H_1}{\geq} \Pr(\mathbf{y}|H_0) \quad \text{(ML Rule)} \tag{40}$$

- MAP and ML minimize Bayesian risk and probability of error.

Simple example

- Assume $y_k = m_0 + v_k$ under hypothesis H_0 and $y_k = m_1 + v_k$ under hypothesis H_1, where $m_1 > m_0$ and variables $\{v_k\}$, $k \in \{1, 2, \ldots, M\}$ are derived from white Gaussian noise (WGN), i.e. $v_i \perp v_j, i \neq j$ (statistically independent) and $v_k \sim \mathcal{N}(0, \sigma^2)$. Find optimal decision rule that detects which hypothesis holds.
- Solution: Affine transformation of Gaussian is also Gaussian:

$$y_k \sim \begin{cases} \mathcal{N}(m_0, \sigma^2), & \text{under} \quad H_0 \\ \mathcal{N}(m_1, \sigma^2), & \text{under} \quad H_1 \end{cases} \tag{41}$$

Since $\{v_k\}$ are independent, observations $\{y_k\}$ are independent and the product of their conditional pdfs offers the conditional probability density of each hypothesis and their ratio:

$$f_{\mathbf{y}|H_j}(\mathbf{y} = [y_1 \;\; y_2 \;\; \ldots y_M]|\mathrm{H}_j) = \prod_{k=1}^{M} f_{y_k|H_j}(y_k|\mathrm{H}_j) = \prod_{k=1}^{M} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_k - m_j)^2}{2\sigma^2}\right]$$

$$= \frac{1}{(2\pi\sigma^2)^{\frac{M}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{k=1}^{M}(y_k - m_j)^2\right].$$

$$L(\mathbf{y}) \triangleq \frac{f_{\mathbf{y}|H_1}(\mathbf{y}|\mathrm{H}_1)}{f_{\mathbf{y}|H_0}(\mathbf{y}|\mathrm{H}_0)} = \exp\left[-\frac{1}{2\sigma^2}\sum_{k=1}^{M}(y_k - m_1)^2 + \frac{1}{2\sigma^2}\sum_{k=1}^{M}(y_k - m_0)^2\right].$$

$$
\begin{aligned}
L(\mathbf{y}) &= \exp\Big[-\frac{1}{2\sigma^2}\sum_{k=1}^{M}(y_k - m_1)^2 + \frac{1}{2\sigma^2}\sum_{k=1}^{M}(y_k - m_0)^2\Big] && (42)\\
&= \exp\Big[-\frac{1}{2\sigma^2}\sum_{k=1}^{M}(y_k^2 - 2\,m_1\,y_k + m_1^2) + \frac{1}{2\sigma^2}\sum_{k=1}^{M}(y_k^2 - 2\,m_0\,y_k + m_0^2)\Big] && (43)\\
&= \exp\Big[+\frac{m_1}{\sigma^2}\sum_{k=1}^{M} y_k - \frac{1}{2\sigma^2}\sum_{k=1}^{M} m_1^2 - \frac{m_0}{\sigma^2}\sum_{k=1}^{M} y_k + \frac{1}{2\sigma^2}\sum_{k=1}^{M} m_0^2\Big] && (44)\\
&= \exp\Big[+\frac{m_1 - m_0}{\sigma^2}\sum_{k=1}^{M} y_k - \frac{1}{2\sigma^2}\sum_{k=1}^{M} m_1^2 + \frac{1}{2\sigma^2}\sum_{k=1}^{M} m_0^2\Big] && (45)\\
&= \exp\Big[+\frac{m_1 - m_0}{\sigma^2}\sum_{k=1}^{M} y_k - \frac{1}{2\sigma^2}\,M\,m_1^2 + \frac{1}{2\sigma^2}\,M\,m_0^2\Big] && (46)\\
&= \exp\Big[-\frac{M\,(m_1^2 - m_0^2)}{2\sigma^2} + \frac{m_1 - m_0}{\sigma^2}\sum_{k=1}^{M} y_k\Big] && (47)
\end{aligned}
$$

$$L(\mathbf{y}) = \exp\Big[-\frac{M\,(m_1^2 - m_0^2)}{2\sigma^2} + \frac{m_1 - m_0}{\sigma^2}\sum_{k=1}^{M} y_k\Big] \tag{48}$$

$$L(\mathbf{y}) \overset{H_1}{\geq} \tau \Leftrightarrow \ln\left(L(\mathbf{y})\right) \overset{H_1}{\geq} \ln(\tau) \Leftrightarrow \tag{49}$$

$$-\frac{M\,(m_1^2 - m_0^2)}{2\sigma^2} + \frac{m_1 - m_0}{\sigma^2}\sum_{k=1}^{M} y_k \overset{H_1}{\geq} \ln(\tau) \Leftrightarrow \tag{50}$$

$$+\frac{m_1 - m_0}{\sigma^2}\sum_{k=1}^{M} y_k \overset{H_1}{\geq} +\frac{M\,(m_1^2 - m_0^2)}{2\sigma^2} + \ln(\tau) \overset{m_1 > m_0}{\Leftrightarrow} \tag{51}$$

$$+\frac{1}{M}\sum_{k=1}^{M} y_k \overset{H_1}{\geq} +\frac{(m_1 + m_0)}{2} + \frac{\sigma^2}{M(m_1 - m_0)}\ln(\tau), \tag{52}$$

where we used the fact that $(m_1 - m_0) > 0$.

- The left-hand side of the above inequality, i.e., the term $\frac{1}{M}\sum_{k=1}^{M} y_k$, is called the sufficient statistic.
- Observe that the sufficient statistic is the sample mean for $M \to +\infty$, under each hypothesis.

References

Bernard C. Levy, Principles of Signal Detection and Parameter Estimation, Springer 2008.

Thank you!

Detection & Estimation Theory: Lecture 3

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Outline

- One more simple (binary hypothesis testing) example
- Common distributions for Sufficient Statistics
- Gaussian vectors (or jointly Gaussian random variables)

Another simple example

- Assume $y_k = v_k$, where $v_k \sim \mathcal{N}(0, \sigma_0^2)$ under hypothesis H_0 and $y_k = v_k$, where $v_k \sim \mathcal{N}(0, \sigma_1^2)$ under hypothesis H_1, with $\sigma_1^2 > \sigma_0^2$ and variables $\{v_k\}$, $k \in \{1, 2, \dots, M\}$ are derived from white Gaussian noise (WGN), i.e. $v_i \perp v_j, i \neq j$ (statistically independent) and v_k is Gaussian. Find optimal decision rule that detects which hypothesis holds.
- Notice that in this example, the variance of measurements changes per hypothesis (and not the mean, as in the previous example).
- Solution: Affine transformation of Gaussian is also Gaussian:

$$y_k \sim \begin{cases} \mathcal{N}(0, \sigma_0^2), & \text{under} \quad H_0 \\ \\ \mathcal{N}(0, \sigma_1^2), & \text{under} \quad H_1 \end{cases} \tag{1}$$

Since $\{v_k\}$ are independent, observations $\{y_k\}$ are independent and the product of their conditional pdfs offers the conditional probability density of each hypothesis and their ratio, as follows (with $j \in \{0, 1\}$):

$$f_{\mathbf{y}|H_j}(\mathbf{y} = [y_1 \ y_2 \ \dots y_M]|\mathrm{H}_j) = \prod_{k=1}^{M} f_{y_k|H_j}(y_k|\mathrm{H}_j) = \prod_{k=1}^{M} \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left[-\frac{y_k^2}{2\sigma_j^2}\right]$$

$$= \frac{1}{(2\pi\sigma_j^2)^{\frac{M}{2}}} \exp\Big[-\frac{1}{2\sigma_j^2} \sum_{k=1}^{M} y_k^2 \Big].$$

$$L(\mathbf{y}) \triangleq \frac{f_{\mathbf{y}|H_1}(\mathbf{y}|\mathrm{H}_1)}{f_{\mathbf{y}|H_0}(\mathbf{y}|\mathrm{H}_0)} = \frac{\sigma_0^M}{\sigma_1^M} \exp\Big[\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2} \right) \sum_{k=1}^{M} y_k^2 \Big].$$

$$L(\mathbf{y}) = \frac{\sigma_0^M}{\sigma_1^M} \exp\Big[\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2} \right) \sum_{k=1}^{M} y_k^2 \Big]. \tag{2}$$

$$L(\mathbf{y}) \overset{H_1}{\geq} \tau \Leftrightarrow \ln\left(L(\mathbf{y})\right) \overset{H_1}{\geq} \ln(\tau) \Leftrightarrow \tag{3}$$

$$-M \ln\left(\frac{\sigma_1}{\sigma_0}\right) + \frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2} \sum_{k=1}^{M} y_k^2 \overset{H_1}{\geq} \ln(\tau) \Leftrightarrow \tag{4}$$

$$\frac{1}{M} \; \frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2} \sum_{k=1}^{M} y_k^2 \overset{H_1}{\geq} \frac{1}{M} \; \ln(\tau) + \ln\left(\frac{\sigma_1}{\sigma_0}\right) \overset{\sigma_1^2 > \sigma_0^2}{\Leftrightarrow} \tag{5}$$

$$\frac{1}{M} \sum_{k=1}^{M} y_k^2 \overset{H_1}{\geq} \frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2} \left(\frac{1}{M} \; \ln(\tau) + \ln\left(\frac{\sigma_1}{\sigma_0}\right)\right), \tag{6}$$

where we used the fact in Eq. (6) that $(\sigma_1^2 - \sigma_0^2) > 0$.

- In this case, the sufficient statistic is the term $\frac{1}{M} \sum_{k=1}^{M} y_k^2$.
- Observe that the sufficient statistic is the sample variance (since $\mathbb{E}[y_k] = 0$) for $M \to +\infty$, under each hypothesis (and not the sample mean, as in the previous example).

$$S \triangleq \frac{1}{M} \sum_{k=1}^{M} y_k^2 \overset{H_1}{\geq} \frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2} \left(\frac{1}{M} \ln(\tau) + \ln\left(\frac{\sigma_1}{\sigma_0}\right)\right). \tag{7}$$

Additional remarks:

1. Under H_j, for $\lim_{M\to+\infty} S = \sigma_j^2,\ j \in \{0,1\}$.
2. Using $1 - \frac{1}{x} \leq \ln(x) \leq x - 1$, it can be easily shown that $\sigma_0^2 \leq \frac{\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2} \ln\left(\frac{\sigma_1}{\sigma_0}\right)^2 \leq \sigma_1^2$.
3. For $M \to +\infty$, $\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2} \left(\frac{1}{M} \ln(\tau) + \ln\left(\frac{\sigma_1}{\sigma_0}\right)\right) \to \frac{\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2} \ln\left(\frac{\sigma_1}{\sigma_0}\right)^2$.
4. Under hypothesis H_j, $\frac{M\ S}{\sigma_j^2} = \sum_{k=1}^{M} \left(\frac{y_k}{\sigma_j}\right)^2$ = sum of independent squared zero-mean Gaussians of unit variance: Under hypothesis H_j, $\frac{M\ S}{\sigma_j^2}$ corresponds to Chi-squared distribution with M degrees of freedom [will explain it subsequently].

Useful distributions

- $z = \sum_{i=1}^{M} z_i^2, z_i \sim \mathcal{N}(0,1)$ and $\{z_i\}$ independent, identically distributed (i.i.d.):
  - z distributed according to the Chi-squared distribution with M degrees of freedom and pdf as follows:

$$f_z(z) = \frac{1}{\Gamma(M/2)\, 2^{M/2}}\, z^{\left(\frac{M}{2}-1\right)}\, e^{-z/2}\, u(z), \tag{8}$$

    with $u(z)$ the step function (i.e., $u(z) = 1$ for $z \geq 0$ and zero otherwise) and $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$ Euler's gamma function, defined everywhere apart from non-positive integers (and $\Gamma(n) = (n-1)!$ for any positive integer n).
  - $\mathbb{E}[z] = M$, $\sigma_z^2 \stackrel{\triangle}{=}$ variance of $z = \mathrm{var}(z) = 2M$.
  - Special case - $M = 2$: exponential distribution, with p.d.f as follows (since $\Gamma(M = 2/2) = \Gamma(1) = 0! = 1$):

$$f_z(z) = \frac{1}{2} e^{-z/2}\, u(z). \tag{9}$$

Useful distributions

- In general, the pdf of a random variable according to the exponential distribution with parameter $\lambda > 0$ is given by:

$$f_z(z) = \lambda\, e^{-\lambda z}\, u(z), \tag{10}$$

with $\mathbb{E}[z] = 1/\lambda$ and $\sigma_z^2 \stackrel{\triangle}{=} \mathrm{var}(z) = 1/\lambda^2$.

- This is equivalent to $z = z_1^2 + z_2^2$, with z_1, z_2 independent and identically distributed according to $\mathcal{N}\left(0, \sigma^2\right)$ and $\mathbb{E}[z] = 1/\lambda = 2\sigma^2$.
  - For the special case of $y = \sqrt{z_1^2 + z_2^2}$, the Rayleigh pdf occurs:

$$f_y(y) = \frac{y}{\sigma^2}\, \exp\left(-\frac{y^2}{2\sigma^2}\right) u(y), \tag{11}$$

with $\mathbb{E}[y] = \sigma\sqrt{\pi/2}$ and $\sigma_y^2 = \mathrm{var}(y) = \frac{4-\pi}{2}\sigma^2$.

Useful distributions

- Set $z = \sum_{i=1}^{2M} z_i^2, z_i \sim \mathcal{N}(0,1)$ and $\{z_i\}$ i.i.d; as explained, z is distributed according to the Chi-squared distribution with $2M$ degrees of freedom.
- What is the distribution of $x = \theta\, z$ $(\theta > 0)$?
  - x can be viewed as the sum of M i.i.d. random variables distributed each according to the exponential distribution with parameter $\lambda = 1/(2\theta)$.
  - for any differentiable and invertible function $x = g(z)$, we do know that the new pdf can be found as follows:

$$f_x(x) = \frac{f_z(z)}{|g'(z)|}\Big|_{z=g^{-1}(x)} \tag{12}$$

and thus,

$$f_x(x) = \frac{f_z(z)}{\theta}\Big|_{z=x/\theta} = \frac{1}{(2\theta)^M\,\Gamma(M)}\, x^{(M-1)} \exp\left(-\frac{x}{2\theta}\right) u(x),$$

which corresponds to the Gamma distribution $(\Gamma(M, 2\theta))$, with parameters $M,\ 2\theta$ and $\mathbb{E}[x] = 2M\theta$, $\mathrm{var}(x) = 4M\theta^2$.

Useful distributions

- Set $z = \sum_{i=1}^{2M} z_i^2, z_i \sim \mathcal{N}(0,1)$ and $\{z_i\}$ i.i.d; as explained, z is distributed according to the Chi-squared distribution with $2M$ degrees of freedom.
- The pdf of $x = \theta\, z$ $(\theta > 0)$ is the pdf of the sum of M i.i.d. exponentials:

$$f_x(x) = \frac{1}{(2\theta)^M\,\Gamma(M)}\, x^{(M-1)} \exp\left(-\frac{x}{2\theta}\right) u(x),$$

  which corresponds, as shown, to the Gamma distribution $\Gamma(M, 2\theta)$, with parameters $M,\ 2\theta$ and $\mathbb{E}[x] = 2M\theta$, $\mathrm{var}(x) = 4M\theta^2$.
- Other distributions of the exponential family to remember:
  - (discrete) Poisson: $\Pr(n) = \frac{\lambda}{n!}\, e^{\lambda}$, $n \in \mathbb{N}$, $\mathbb{E}[n] = \lambda = \mathrm{var}(n)$.
  - (continuous) Laplace: $f_x(x;\mu,\beta) = \frac{1}{2\beta} \exp\left(-\frac{|x-\mu|}{\beta}\right)$, $\mathbb{E}[x] = \mu, \mathrm{var}(x) = 2\beta^2$.

Gaussian vectors (or jointly Gaussian random variables)

- Let $\mathbf{x} = [x_1 \ x_2 \ \ldots \ x_M]^{\mathrm{T}}$, where $x_1, \ x_2, \ \ldots, \ x_M$ are real. Vector $\mathbf{x}$ is Gaussian, or equivalently, $x_1 \ x_2 \ \ldots \ x_M$ are jointly Gaussian, if and only if the pdf of $\mathbf{x}$ (or the joint pdf of $x_1 \ x_2 \ \ldots \ x_M$) is given as follows:

  Covariance form:

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^M \ |\boldsymbol{\Sigma}|}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\mathbf{m})\right\} \tag{13}$$

  denoted as $\mathbf{x} \sim \mathcal{N}(\mathbf{m}, \boldsymbol{\Sigma})$, with mean $\mathbf{m} \stackrel{\triangle}{=} \mathbb{E}[\mathbf{x}]$, covariance matrix $\boldsymbol{\Sigma} \stackrel{\triangle}{=} \mathbb{E}\left[(\mathbf{x}-\mathbf{m})(\mathbf{x}-\mathbf{m})^{\mathrm{T}}\right]$ and $|\boldsymbol{\Sigma}|$ is the determinant of $\boldsymbol{\Sigma}$.

Gaussian vectors (or jointly Gaussian random variables)

- Let $\mathbf{x} = [x_1 \ x_2 \ \ldots \ x_M]^{\mathrm{T}}$, where $x_1, \ x_2, \ \ldots, \ x_M$ are real. Vector $\mathbf{x}$ is Gaussian, or equivalently, $x_1 \ x_2 \ \ldots \ x_M$ are jointly Gaussian, if and only if the pdf of $\mathbf{x}$ (or the joint pdf of $x_1 \ x_2 \ \ldots \ x_M$) is given as follows:

  Information form:
  $$f_{\mathbf{x}}(\mathbf{x}) \propto \exp\left\{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{J}\mathbf{x} + \mathbf{h}^{\mathrm{T}}\mathbf{x}\right\}, \tag{14}$$

  denoted as $\mathbf{x} \sim \mathcal{N}^{-1}(\mathbf{h}, \mathbf{J})$, with potential vector $\mathbf{h} = \mathbf{J}\,\mathbf{m}$ and information (or precision) matrix $\mathbf{J} = \mathbf{\Sigma}^{-1}$.

Gaussian vector properties

- Let $\mathbf{x}$ (real) Gaussian vector. The following hold:
  - Moment generating function (MGF) $M_{\mathbf{x}}(j\mathbf{u})$:

$$M_{\mathbf{x}}(j\mathbf{u}) \triangleq \mathbb{E}\left[e^{j\mathbf{u}^{\mathrm{T}}\mathbf{x}}\right] = \exp\left\{j\mathbf{u}^{\mathrm{T}}\mathbf{m} - \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{u}\right\}. \tag{15}$$

  - All linear combinations of elements of $\mathbf{x}$ are scalar Gaussian random variables: $y = \mathbf{a}^{\mathrm{T}}\mathbf{x}$ is Gaussian for all deterministic $\mathbf{a}$.
  - There exists deterministic matrix $\mathbf{A}$, deterministic vector vector $\mathbf{b}$ and random vector $\mathbf{v}$ of i.i.d. $\mathcal{N}(0,1)$ entries, such that $\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{b}$.
  - Affine transformation is also Gaussian, i.e., for any deterministic matrix $\mathbf{A}$ and deterministic vector $\mathbf{b}$, random vector $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$ is Gaussian, according to $\mathcal{N}\left(\mathbf{A}\mathbf{m} + \mathbf{b}, \mathbf{A}\mathbf{\Sigma}\mathbf{A}^{\mathrm{T}}\right)$. [Simple proof from MGF]

Gaussian vector properties

- Let $\mathbf{x} = [\mathbf{y}^{\mathrm{T}}\ \mathbf{z}^{\mathrm{T}}]^{\mathrm{T}}$ (real) Gaussian vector, where $\mathbf{y}$, $\mathbf{z}$ are (real) vectors of appropriate dimensions. The following properties hold:
  - $\mathbf{y}$ is Gaussian.
  - $\mathbf{z}$ is Gaussian.
  - $\mathbf{y}$ given $\mathbf{z}$ is Gaussian.
  - $\mathbf{z}$ given $\mathbf{y}$ is Gaussian.
  - $\mathbb{E}[\mathbf{y}|\mathbf{z}]$ = affine transformation of $\mathbf{z} \Rightarrow$ Gaussian.
  - $\mathbb{E}[\mathbf{y}\ \mathbf{z}^{\mathrm{T}}] = \mathbb{E}[\mathbf{y}]\ \mathbb{E}[\mathbf{z}]^{\mathrm{T}} \Rightarrow \mathbf{y} \perp \mathbf{z}$, i.e., jointly Gaussian and uncorrelated results to independent!
- However, even if $\mathbf{y}$ is Gaussian and $\mathbf{z}$ is Gaussian, $\mathbf{x} = [\mathbf{y}^{\mathrm{T}}\ \mathbf{z}^{\mathrm{T}}]^{\mathrm{T}}$ may not be Gaussian. In other words, $\mathbf{y}$ and $\mathbf{z}$ may not be necessarily *jointly* Gaussian!

- Counterexample: let $x,\ y$ jointly Gaussian, zero mean, scalar random variables with joint pdf as follows:

$$p_{x,y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\ \exp\left\{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)\right\}, \tag{16}$$

which corresponds to the Gaussian vector $[x\ \ y]^{\mathrm{T}}$, with $\mathbf{m} = [0\ \ 0]^{\mathrm{T}}$ and $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$.
- Clearly $p_y(y) = \int_{-\infty}^{+\infty} p_{x,y}(x,y)dx = 2\int_0^{+\infty} p_{x,y}(x,y)dx$ corresponding to $\mathcal{N}(0,\sigma_y^2)$.
- Set the following non-Gaussian pdf:

$$\hat{p}_{x,y}(x,y) = \begin{cases} \frac{1}{\pi\sigma_x\sigma_y}\ \exp\left\{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)\right\}, & \text{if } x\ y > 0, \\ 0, & \text{if } x\ y < 0. \end{cases}$$

In this case, for $y > 0$, $\int_{-\infty}^{+\infty} \hat{p}_{x,y}(x,y)dx = \int_0^{+\infty} \hat{p}_{x,y}(x,y)dx = 2\ \int_0^{+\infty} \hat{p}_{x,y}(x,y)dx = p_y(y)$, i.e., Gaussian.

References

[1] Bernard C. Levy, Principles of Signal Detection and Parameter Estimation, Springer 2008.
[2] Class instructor notes.

Thank you!

Detection & Estimation Theory: Lecture 4

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Outline

- Probabilities that fully characterise a test: P_D vs P_F
- Neyman-Pearson Test
  - Derivation

Probability of Detection vs Probability of False Alarm

- There are two probability metrics for test $\delta(\mathbf{y})$ that fully characterise a binary hypothesis testing problem (as well as any binary classifier):
  - Probability of detection $\mathrm{P_D}$:

$$\mathrm{P_D}(\delta) \triangleq \int_{\mathcal{Y}_1} f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1)d\mathbf{y}. \tag{1}$$

    Complementary to the above, Probabilty of a miss is defined as $\mathrm{P_M}(\delta) \triangleq 1 - \mathrm{P_D}(\delta) = \int_{\mathcal{Y}_0} f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1)d\mathbf{y}$.
  - Probability of false alarm $\mathrm{P_F}$:

$$\mathrm{P_F}(\delta) \triangleq \int_{\mathcal{Y}_1} f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0)d\mathbf{y}. \tag{2}$$

- Notice that both are calculated over $\mathcal{Y}_1$, i.e., for space of measurements where decision is $\delta(\mathbf{y}) = 1$.
- Think of a radar system: false alarm is when the radar reports an airplane is coming, when it is not.

Probability of Detection vs Probability of False Alarm

- Remember the Bayes conditional risk:

  $$R(\delta(\mathbf{y})|\mathrm{H}_j) = C_{1j}\ \Pr(\delta(\mathbf{y}) = 1|\mathrm{H}_j) + C_{0j}\ \Pr(\delta(\mathbf{y}) = 0|\mathrm{H}_j)$$

  Thus,

  $$\begin{aligned} R(\delta(\mathbf{y})|\mathrm{H}_0) &= C_{10}\ \Pr(\delta(\mathbf{y}) = 1|\mathrm{H}_0) + C_{00}\ \Pr(\delta(\mathbf{y}) = 0|\mathrm{H}_0) \\ &= C_{10}\ \mathrm{P_F} + C_{00}\ (1 - \mathrm{P_F}). \end{aligned} \tag{3}$$

  and similarly,

  $$\begin{aligned} R(\delta(\mathbf{y})|\mathrm{H}_1) &= C_{11}\ \Pr(\delta(\mathbf{y}) = 1|\mathrm{H}_1) + C_{01}\ \Pr(\delta(\mathbf{y}) = 0|\mathrm{H}_1) \\ &= C_{11}\ \mathrm{P_D} + C_{01}\ (1 - \mathrm{P_D}). \end{aligned} \tag{4}$$

- Thus, the (unconditional) Bayes risk of test δ is fully characterised by the pair $(\mathrm{P_F}, \mathrm{P_D})$ of specific test δ:

  $$\begin{aligned} R(\delta) &= R(\delta(\mathbf{y})|\mathrm{H}_0)\,\pi_0 + R(\delta(\mathbf{y})|\mathrm{H}_1)\,\pi_1 \\ &= C_{00}\,\pi_0 + C_{01}\,\pi_1 + \pi_0\,(C_{10} - C_{00})\ \mathrm{P_F} + \pi_1\,(C_{11} - C_{01})\,\mathrm{P_D} \end{aligned} \tag{5}$$

Probability of Detection vs Probability of False Alarm

- The (unconditional) Bayes risk of test δ is fully characterised by the pair $(\mathrm{P_F}, \mathrm{P_D})$ of specific test δ:

$$
\begin{aligned}
R(\delta) &= R(\delta(\mathbf{y})|\mathrm{H}_0)\,\pi_0 + R(\delta(\mathbf{y})|\mathrm{H}_1)\,\pi_1 \\
&= C_{00}\,\pi_0 + C_{01}\,\pi_1 + \;\pi_0\,(C_{10} - C_{00})\,\mathrm{P_F} + \pi_1\,(C_{11} - C_{01})\,\mathrm{P_D}
\end{aligned} \tag{6}
$$

- Ideally, we would like to have a test (or a binary classifier) with $(\mathrm{P_F}, \mathrm{P_D}) = (0, 1)$. However, this is not feasible!
- Next lecture will offer the feasible pairs $(\mathrm{P_F}, \mathrm{P_D})$, as well as properties of the boundary between feasible and non-feasible pairs for all tests!
- Boundary between feasible and non-feasiible tests: receiver operating characteristic (ROC) [next lecture].

Neyman-Pearson Test

- Problem definition: among the tests that bound false alarm probability, find the test that maximises probability of detection. The formulation is given below:

$$\delta_{\mathrm{NP}} = \arg\max_{\delta \in D_\alpha} \mathrm{P_D}(\delta) \tag{7}$$

$$D_\alpha = \{\text{all tests } \delta : \mathrm{P_F}(\delta) \leq \alpha\}$$

- The problem could be also formulated with bounded probability of detection and minimised probability of false alarm.

Neyman-Pearson Test Derivation

- Constrained optimization problem: we need to set Lagrangian and KKT condition(s).
  - Lagrangian $L(\cdot,\cdot)$ for Lagrange multiplier $\lambda \geq 0$:

$$\begin{aligned} L(\delta,\lambda) &= \mathrm{P_D}(\delta) + \lambda\,(\alpha - \mathrm{P_F}(\delta)) \\ &= \lambda\,\alpha + \int_{\mathcal{Y}_1} \left[f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1) - \lambda\, f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0) \right] d\mathbf{y} \end{aligned} \tag{8}$$

  - KKT condition ($\lambda \geq 0$):

$$\lambda\,(\alpha - \mathrm{P_F}(\delta)) = 0 \tag{9}$$

- Optimal test maximizes Lagrangian in Eq. (10) AND also satisfies KKT condition in Eq. (9).

Neyman-Pearson Test Derivation

- Maximization of Lagrangian $L(\delta,\lambda)(\lambda \geq 0)$:

$$L(\delta,\lambda) = \lambda\,\alpha + \int_{\mathcal{Y}_1} \left[f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1) - \lambda\, f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0) \right] d\mathbf{y} \quad (10)$$

  - if $\mathbf{y} \in \mathcal{Y}_1$ then $\left[f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1) - \lambda\, f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0) \right] > 0$, otherwise the Lagrangian is not maximized.
  - Equivalently, if $\mathbf{y} \in \mathcal{Y}_0$, then $\left[f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1) - \lambda\, f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0) \right] < 0$ and that particular $\mathbf{y}$ is not taken into account in Eq. (10).
  - for any $\mathbf{y}$ with $\left[f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1) - \lambda\, f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0) \right] = 0$, what decision should we adopt?
- Thus, maximization of the Lagrangian results to testing the sign of $\left[f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1) - \lambda\, f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0) \right]$.

Neyman-Pearson Test Derivation

- Maximization of the Lagrangian results to testing the sign of $\left[f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1) - \lambda\, f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0)\right]$. Thus, optimal test for given $\lambda \geq 0$ follows:

$$\delta(\mathbf{y}) = \begin{cases} 1, & f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1) - \lambda\, f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0) > 0 \\ 0, & f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1) - \lambda\, f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0) < 0 \\ 0 \text{ or } 1 \text{ (TBD)}\ , & f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1) - \lambda\, f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0) = 0. \end{cases} \tag{11}$$

- Setting the likelihood ratio $L(\mathbf{y}) \stackrel{\triangle}{=} f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1)/f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0)$, Eq. (11) is equivalent to:

$$\delta(\mathbf{y}) = \begin{cases} 1, & L(\mathbf{y}) > \lambda, \\ 0, & L(\mathbf{y}) < \lambda, \\ 0 \text{ or } 1 \text{ (TBD)}\ , & L(\mathbf{y}) = \lambda. \end{cases} \tag{12}$$

Neyman-Pearson Test Derivation

- Maximization of the Lagrangian results to the following optimal test for given $\lambda \geq 0$:

$$\delta(\mathbf{y}) = \begin{cases} 1, & L(\mathbf{y}) > \lambda, \\ 0, & L(\mathbf{y}) < \lambda, \\ 0 \text{ or } 1 \text{ (TBD)}, & L(\mathbf{y}) = \lambda. \end{cases} \tag{13}$$

- We need to find out λ and decision for $L(\mathbf{y}) = \lambda$.
- We define the following conditional cumulative distribution function (cdf):

$$F_{\mathrm{L}}\left(l|\mathrm{H}_0\right) \triangleq \Pr\left(L(\mathbf{y}) \leq l|\mathrm{H}_0\right). \tag{14}$$

  - As any cdf, the above should be:
    1. right-continuous,
    2. non-decreasing with increasing l,
    3. 0 for $l \to -\infty$ and
    4. 1 for $l \to +\infty$.

Neyman-Pearson Test Derivation

- Any cdf is "right-continuous": check figure below!

$$F_{\mathrm{L}}\left(l|\mathrm{H}_0\right) \triangleq \Pr\left(L(\mathbf{y}) \leq l|\mathrm{H}_0\right). \qquad (15)$$

- As any cdf, the above should be:
  1. right-continuous,
  2. non-decreasing with increasing l,
  3. 0 for $l \to -\infty$ and
  4. 1 for $l \to +\infty$.

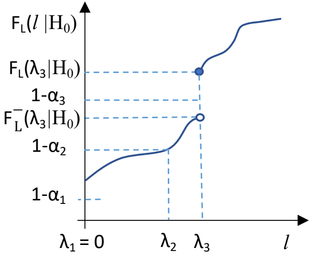
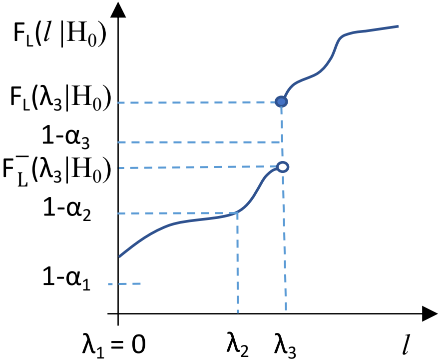


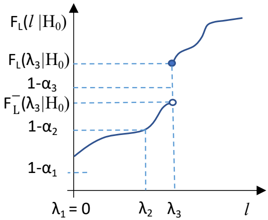
Figure 1: Example cdf with right-continuity.

Neyman-Pearson Test Derivation



- Three cases of likelihood ratio threshold λ occur, depending on value of $1 - \alpha$ vs $F_{\mathrm{L}}\left(l = 0|\mathrm{H}_0\right)$.
- Reminder: α is the upper bound of $\mathrm{P_F}$.

Neyman-Pearson Test Derivation

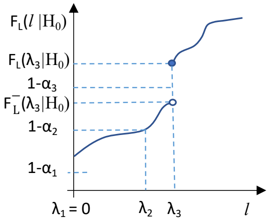


- Case I: $1 - \alpha < F_L\,(l = 0|H_0) \overset{\triangle}{=} f_0 \Leftrightarrow 1 - f_0 < \alpha$:
  - set $\lambda = 0$ and $\delta(\mathbf{y}) = 0$ for $L(\mathbf{y}) = \lambda = 0$, i.e.,

$$\delta(\mathbf{y}) = \begin{cases} 1, & L(\mathbf{y}) > \lambda = 0, \\ 0, & L(\mathbf{y}) \leq \lambda = 0. \end{cases} \tag{16}$$

  - KTT $\lambda\,(\alpha - P_F) = 0$ is satisfied for $\lambda = 0$.
  - $P_F = 1 - \Pr\,(\delta(\mathbf{y}) = 0|H_0) = 1 - \Pr\,(L(\mathbf{y}) \leq 0|H_0) = 1 - f_0 < \alpha \Rightarrow$ probability of false alarm is bounded.

Neyman-Pearson Test Derivation

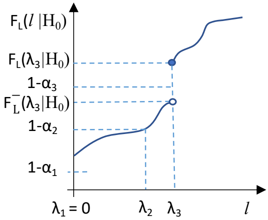


- Case II: $1-\alpha \geq F_{\mathrm{L}}(l=0|\mathrm{H}_0) \stackrel{\triangle}{=} f_0$ and λ^* is in the range of $F_{\mathrm{L}}(l|\mathrm{H}_0)$, i.e., there is λ^* such that $F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0) = 1-\alpha$.
  - set $\lambda = \lambda^*$ and $\delta(\mathbf{y}) = 0$ for $L(\mathbf{y}) = \lambda^*$, i.e.,

$$\delta(\mathbf{y}) = \begin{cases} 1, & L(\mathbf{y}) > \lambda^*, \\ 0, & L(\mathbf{y}) \leq \lambda^*. \end{cases} \tag{17}$$

  - $\mathrm{P_F} = 1 - \Pr(\delta(\mathbf{y}) = 0|\mathrm{H}_0) = 1 - \Pr(L(\mathbf{y}) \leq \lambda^*|\mathrm{H}_0) = 1-(1-\alpha) = \alpha$.
  - KTT $\lambda(\alpha - \mathrm{P_F}) = 0$ is satisfied for $\lambda = \lambda^*$.

Neyman-Pearson Test Derivation

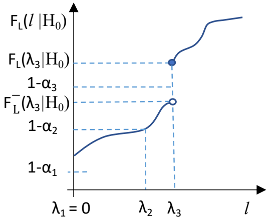


- Case III: $1-\alpha \geq F_{\mathrm{L}}\left(l=0|\mathrm{H}_0\right) \triangleq f_0$ and λ^* is NOT in the range of $F_{\mathrm{L}}\left(l|\mathrm{H}_0\right)$, i.e., $F_{\mathrm{L}}^-\left(\lambda^*|\mathrm{H}_0\right) < 1-\alpha < F_{\mathrm{L}}\left(\lambda^*|\mathrm{H}_0\right)$.
  - set $\lambda = \lambda^*$ and $\delta(\mathbf{y}) = 0$ for $L(\mathbf{y}) = \lambda$, i.e.,

$$\delta(\mathbf{y}) \triangleq \delta_{L,\lambda^*}(\mathbf{y}) = \begin{cases} 1, & L(\mathbf{y}) > \lambda^*, \\ 0, & L(\mathbf{y}) \leq \lambda^*. \end{cases} \tag{18}$$

  - $\mathrm{P_F} = 1 - \Pr\left(\delta(\mathbf{y}) = 0|\mathrm{H}_0\right) = 1 - \Pr\left(L(\mathbf{y}) \leq \lambda^*|\mathrm{H}_0\right) = 1 - F_{\mathrm{L}}\left(\lambda^*|\mathrm{H}_0\right) < \alpha$.
  - KTT $\lambda\left(\alpha - \mathrm{P_F}\right) = 0$ is NOT satisfied!

Neyman-Pearson Test Derivation

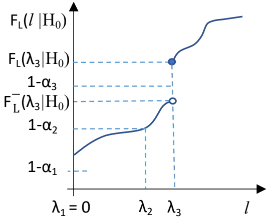


- Case III: $1-\alpha \geq F_L\left(l=0|\mathrm{H}_0\right) \triangleq f_0$ and λ^* is NOT in the range of $F_L\left(l|\mathrm{H}_0\right)$, i.e., $F_L^-\left(\lambda^*|\mathrm{H}_0\right) < 1-\alpha < F_L\left(\lambda^*|\mathrm{H}_0\right)$.
  - set $\lambda = \lambda^*$ and $\delta(\mathbf{y}) = 1$ for $L(\mathbf{y}) = \lambda$, i.e.,

$$\delta(\mathbf{y}) \triangleq \delta_{U,\lambda^*}(\mathbf{y}) = \begin{cases} 1, & L(\mathbf{y}) \geq \lambda^*, \\ 0, & L(\mathbf{y}) < \lambda^*. \end{cases} \tag{19}$$

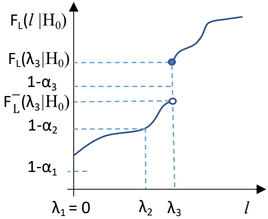
  - $\mathrm{P_F} = 1 - \Pr\left(\delta(\mathbf{y}) = 0|\mathrm{H}_0\right) = 1 - \Pr\left(L(\mathbf{y}) < \lambda^*|\mathrm{H}_0\right) = 1 - F_L^-\left(\lambda^*|\mathrm{H}_0\right) > \alpha$.
  - KTT $\lambda\left(\alpha - \mathrm{P_F}\right) = 0$ is NOT satisfied!

Neyman-Pearson Test Derivation



- Case III: $1-\alpha \geq F_{\mathrm{L}}\left(l=0|\mathrm{H}_0\right) \stackrel{\triangle}{=} f_0$ and λ^* is NOT in the range of $F_{\mathrm{L}}\left(l|\mathrm{H}_0\right)$, i.e., $F_{\mathrm{L}}^{-}\left(\lambda^*|\mathrm{H}_0\right) < 1-\alpha < F_{\mathrm{L}}\left(\lambda^*|\mathrm{H}_0\right)$. So far:
  - Test $\delta_{L,\lambda^*}(\mathbf{y})$ with $\mathrm{P}_{\mathrm{F}}\left(\delta_{L,\lambda^*}\right) < \alpha$.
  - Test $\delta_{U,\lambda^*}(\mathbf{y})$ with $\mathrm{P}_{\mathrm{F}}\left(\delta_{U,\lambda^*}\right) > \alpha$.
  - KTT $\lambda\left(\alpha-\mathrm{P}_{\mathrm{F}}\right)=0$ requires exactly $\mathrm{P}_{\mathrm{F}}=\alpha$.
- Solution?

Neyman-Pearson Test Derivation

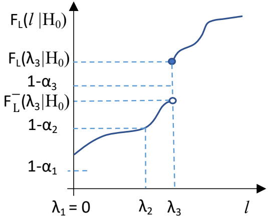


- Case III: $1 - \alpha \geq F_{\mathrm{L}}\left(l = 0|\mathrm{H}_0\right) \stackrel{\triangle}{=} f_0$ and λ^* is NOT in the range of $F_{\mathrm{L}}\left(l|\mathrm{H}_0\right)$, i.e., $F_{\mathrm{L}}^{-}\left(\lambda^*|\mathrm{H}_0\right) < 1 - \alpha < F_{\mathrm{L}}\left(\lambda^*|\mathrm{H}_0\right)$.
  - Solution: set $\lambda = \lambda^*$ and randomize decision for $L(\mathbf{y}) = \lambda$:

$$\delta(\mathbf{y}) = \begin{cases} \delta_{U,\lambda^*}(\mathbf{y}), & \text{with probability } \rho, \\ \delta_{L,\lambda^*}(\mathbf{y}), & \text{with probability } 1-\rho, \end{cases} \tag{20}$$

  - Set $0 < \rho < 1$ such that $\mathrm{P_F} \equiv \alpha$ (and KKT is thus satisfied).

Neyman-Pearson Test Derivation



- Case III: $1-\alpha \geq F_{\mathrm{L}}(l=0|\mathrm{H}_0) \triangleq f_0$ and λ^* is NOT in the range of $F_{\mathrm{L}}(l|\mathrm{H}_0)$, i.e., $F_{\mathrm{L}}^-(\lambda^*|\mathrm{H}_0) < 1-\alpha < F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0)$.
  - Solution: set $\lambda = \lambda^*$ and randomize decision for $L(\mathbf{y}) = \lambda$:

$$\delta(\mathbf{y}) = \begin{cases} \delta_{U,\lambda^*}(\mathbf{y}), & \text{with probability } \rho, \\ \delta_{L,\lambda^*}(\mathbf{y}), & \text{with probability } 1-\rho, \end{cases} \tag{21}$$

$$\rho = \frac{F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0) - (1-\alpha)}{F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0) - F_{\mathrm{L}}^-(\lambda^*|\mathrm{H}_0)}, \; 0 < \rho < 1. \tag{22}$$

  - Such $0 < \rho < 1$ guarantees $\mathrm{P_F} \equiv \alpha$.

Neyman-Pearson Test Derivation

- Case III: $1-\alpha \geq F_{\mathrm{L}}(l=0|\mathrm{H}_0) \stackrel{\triangle}{=} f_0$ and λ^* is NOT in the range of $F_{\mathrm{L}}(l|\mathrm{H}_0)$, i.e., $F_{\mathrm{L}}^{-}(\lambda^*|\mathrm{H}_0) < 1-\alpha < F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0)$.
  - Solution: set $\lambda = \lambda^*$ and randomize decision for $L(\mathbf{y}) = \lambda$:

$$\delta(\mathbf{y}) = \begin{cases} \delta_{U,\lambda^*}(\mathbf{y}), & \text{with probability } \rho, \\ \delta_{L,\lambda^*}(\mathbf{y}), & \text{with probability } 1-\rho, \end{cases} \tag{23}$$

$$\rho = \frac{F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0) - (1-\alpha)}{F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0) - F_{\mathrm{L}}^{-}(\lambda^*|\mathrm{H}_0)},\ 0 < \rho < 1. \tag{24}$$

  - Such ρ guarantees $\mathrm{P_F} \equiv \alpha$.
    Proof:

$$\begin{aligned} \mathrm{P_F} &= \rho\,\mathrm{P_F}(\delta_{U,\lambda^*}) + (1-\rho)\,\mathrm{P_F}(\delta_{L,\lambda^*}) && (25) \\ &= \rho\left(1 - F_{\mathrm{L}}^{-}(\lambda^*|\mathrm{H}_0)\right) + (1-\rho)\,(1 - F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0)) && (26) \\ &= 1 - F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0) + \rho\left(F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0) - F_{\mathrm{L}}^{-}(\lambda^*|\mathrm{H}_0)\right) && (27) \\ &= 1 - F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0) + F_{\mathrm{L}}(\lambda^*|\mathrm{H}_0) - (1-\alpha) && (28) \\ &= \alpha && (29) \end{aligned}$$

Neyman-Pearson Test

- Neyman-Pearson-optimal detector is likelihood ratio test (LRT)!
- As already mentioned, we could have minimized $\mathrm{P_F}$ subject to bounded $\mathrm{P_D}$.
- Next lecture: feasible points $(\mathrm{P_F}, \mathrm{P_D})$ for any test!

References

[1] Bernard C. Levy, Principles of Signal Detection and Parameter Estimation, Springer 2008.
[2] Class instructor notes.

Thank you!

Detection & Estimation Theory: Lectures 5 & 6

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School of Electrical and Computer Engineering

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ΕΣΠΑ
2014-2020
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Outline

- Receiver Operating Characteristic (ROC)
- ROC Properties
- Remarks
- Examples

ROC definition

- Remember the two probability metrics that fully characterise a test $\delta(\mathbf{y})$ (as well as any binary classifier):
  - Prob. of detection $\mathrm{P_D}$ and prob. of false alarm $\mathrm{P_F}$:

$$\mathrm{P_D}(\delta) \triangleq \int_{\mathcal{Y}_1} f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1)d\mathbf{y}, \mathrm{P_F}(\delta) \triangleq \int_{\mathcal{Y}_1} f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0)d\mathbf{y}.$$

- The Bayes risk of test δ is fully characterised by the pair $(\mathrm{P_F}, \mathrm{P_D})$ of specific test δ (previous lecture).
- Which pairs $(\mathrm{P_F}, \mathrm{P_D})$ are feasible?
- Boundary between feasible and non-feasiible tests = receiver operating characteristic (ROC).
- ROC properties?

ROC definition

- Remember the two probability metrics that fully characterise a test $\delta(\mathbf{y})$ (as well as any binary classifier):
  - Prob. of detection $\mathrm{P_D}$ and prob. of false alarm $\mathrm{P_F}$:

$$\mathrm{P_D}(\delta) \triangleq \int_{\mathcal{Y}_1} f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1)d\mathbf{y}, \quad \mathrm{P_F}(\delta) \triangleq \int_{\mathcal{Y}_1} f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0)d\mathbf{y}.$$

- Define likelihood ratio and conditional pdf of likelihood ratio $f_{L|\mathrm{H}_j}(l|\mathrm{H}_j)$:

$$L(\mathbf{y}) \triangleq \frac{f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1)}{f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0)}. \tag{1}$$

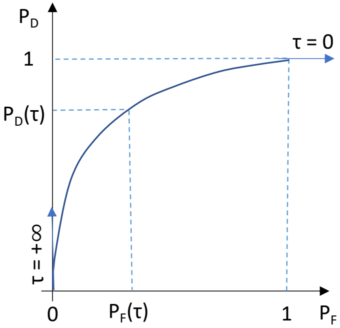
- For likelihood ratio test $L(\mathbf{y}) \overset{H_1}{\geq} \tau$, $\mathrm{P_F}, \mathrm{P_D}$ can be redefined:

$$\mathrm{P_D}(\tau) \triangleq \int_{\tau}^{+\infty} f_{L|\mathrm{H}_1}(l|\mathrm{H}_1)\,dl, \quad \mathrm{P_F}(\tau) \triangleq \int_{\tau}^{+\infty} f_{L|\mathrm{H}_0}(l|\mathrm{H}_0)\,dl.$$

ROC Property 1

- Points $(0, 0)$ and $(1, 1)$ of $(\mathrm{P_F},\ \mathrm{P_D})$ belong to ROC.
  Proof:
  - Always select H_1 (or equivalently, set $\tau = 0$):
  $$\mathrm{P_F}(\tau = 0) = 1,\ \mathrm{P_D}(\tau = 0) = 1. \tag{2}$$
  - Always select H_0 (or equivalently, set $\tau = +\infty$):
  $$\mathrm{P_F}(\tau = +\infty) = 0,\ \mathrm{P_D}(\tau = +\infty) = 0. \tag{3}$$

ROC Property 2



- Slope of ROC at $(\mathrm{P_F}(\tau), \mathrm{P_D}(\tau))$ is equal to τ, i.e., $\frac{d\mathrm{P_D}(\tau)}{d\mathrm{P_F}(\tau)} = \tau$.
  Proof:
  - $\mathrm{P_D}(\delta) = \int_{\tau}^{+\infty} f_{L|\mathrm{H}_1}(l|\mathrm{H}_1)\, d\mathbf{y} = 1 - \int_{-\infty}^{+\tau} f_{L|\mathrm{H}_1}(l|\mathrm{H}_1)\, d\mathbf{y} \Rightarrow$

ROC Property 2

- $\mathrm{P_D}(\tau) = \int_{\tau}^{+\infty} f_{L|\mathrm{H}_1}(l|\mathrm{H}_1)\,dl = 1 - \int_{-\infty}^{+\tau} f_{L|\mathrm{H}_1}(l|\mathrm{H}_1)\,dl \Rightarrow$

$$\frac{d\mathrm{P_D}}{d\tau}(\tau) = -f_{L|\mathrm{H}_1}(\tau|\mathrm{H}_1), \tag{4}$$

$$\text{similarly, } \frac{d\mathrm{P_F}}{d\tau}(\tau) = -f_{L|\mathrm{H}_0}(\tau|\mathrm{H}_0). \tag{5}$$

From Eqs. (4), (5):

$$\frac{d\mathrm{P_D}}{d\mathrm{P_F}}(\tau) = \frac{f_{L|\mathrm{H}_1}(\tau|\mathrm{H}_1)}{f_{L|\mathrm{H}_0}(\tau|\mathrm{H}_0)}. \tag{6}$$

- Recall that:

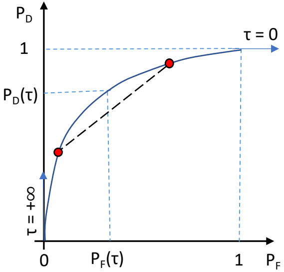
$$\mathrm{P_D}(\tau) = \int_{\tau}^{+\infty} f_{L|\mathrm{H}_1}(l|\mathrm{H}_1)\,dl = \int_{\mathcal{Y}_1} f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1)d\mathbf{y} = \int_{\mathcal{Y}_1} \frac{f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1)}{f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0)} f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0)d\mathbf{y}$$

$$= \int_{\mathcal{Y}_1=\{\mathbf{y}:L(\mathbf{y})\geq\tau\}} L(\mathbf{y})\, f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0)\,d\mathbf{y} = \int_{\tau}^{+\infty} l\cdot f_{L|\mathrm{H}_0}(l|\mathrm{H}_0)\,dl \Rightarrow \tag{7}$$

$$\frac{d\mathrm{P_D}}{d\tau}(\tau) = -\tau\cdot f_{L|\mathrm{H}_0}(\tau|\mathrm{H}_0) \overset{(4)}{\Rightarrow} f_{L|\mathrm{H}_1}(\tau|\mathrm{H}_1) = \tau\, f_{L|\mathrm{H}_0}(\tau|\mathrm{H}_0). \tag{8}$$

From Eqs. (6), (8), the proof is completed.

ROC Property 3



- The domain of feasible points $(\mathrm{P_F}, \mathrm{P_D})$ is convex.
  Proof:
  - We need to show that for any two feasible points $(\mathrm{P_{F1}}, \mathrm{P_{D1}})$, $(\mathrm{P_{F2}}, \mathrm{P_{D2}})$, the line connecting them is also included in the feasible points.
  - Such line is described by $\rho \in [0, 1]$:

$$\mathrm{P_F}(\rho) = \rho\,\mathrm{P_{F1}} + (1-\rho)\,\mathrm{P_{F2}}, \tag{9}$$

$$\mathrm{P_D}(\rho) = \rho\,\mathrm{P_{D1}} + (1-\rho)\,\mathrm{P_{D2}}. \tag{10}$$

ROC Property 3

- Such line is described by $\rho \in [0,1]$:

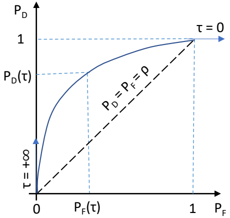
$$\mathrm{P_F}(\rho) = \rho\,\mathrm{P_{F1}} + (1-\rho)\,\mathrm{P_{F2}}, \tag{11}$$
$$\mathrm{P_D}(\rho) = \rho\,\mathrm{P_{D1}} + (1-\rho)\,\mathrm{P_{D2}}. \tag{12}$$

- Why? solve each of the above for ρ and equate: you will find out the line equation connecting the two points.
- Define the following randomized test that selects probabilistic between two tests:

$$\delta(\mathbf{y}) = \begin{cases} \delta_1(\mathbf{y}), & \text{with probability } \rho, \\ \delta_2(\mathbf{y}), & \text{with probability } 1-\rho, \end{cases} \tag{13}$$

  where $\delta_i(\mathbf{y})$ is the feasible test with $\mathrm{P_{F}}_i, \mathrm{P_{D}}_i$, $i \in \{1,2\}$.
- The test above achieves $\mathrm{P_F}(\rho)$ and $\mathrm{P_D}(\rho)$ given in Eqs. (11), (12).

ROC Property 3 Remarks

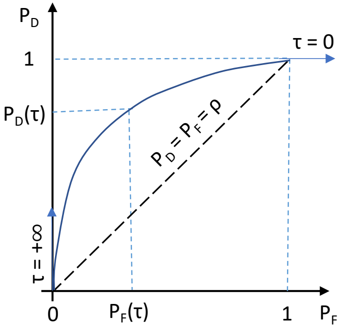


- Define the following randomized test that selects probabilistic between two hypothesis, *independently* of the measurements $\mathbf{y}$:

$$\delta(\mathbf{y}) = \begin{cases} \mathrm{H}_1, & \text{with probability } \rho, \\ \mathrm{H}_0, & \text{with probability } 1-\rho, \end{cases} \tag{14}$$

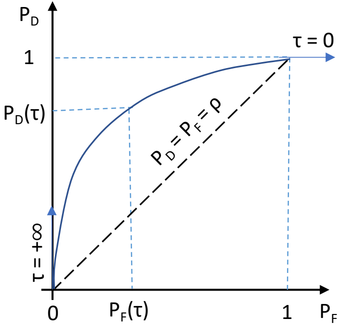
- The test above achieves $\mathrm{P_D}(\rho) = \Pr(\mathcal{Y}_1 | H_1) = \Pr(\mathcal{Y}_1) = \rho$ and $\mathrm{P_F}(\rho) = \Pr(\mathcal{Y}_1 | H_0) = \Pr(\mathcal{Y}_1) = \rho$.
- Thus, line $\mathrm{P_D} = \mathrm{P_F} = \rho$ belongs to the feasible points.

ROC Property 3 Remarks



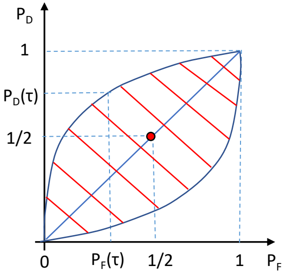
- The domain of feasible points $(\mathrm{P_F}, \mathrm{P_D})$ is convex.
- Line $\mathrm{P_D} = \mathrm{P_F} = \rho$ belongs to the feasible points.
  - ...thus, domain of feasible points are located *below* the ROC curve!
- Domain of feasible points is convex and located below ROC.
  - ...thus, ROC curve is concave!

ROC Property 4



- For tests on the ROC curve, $\mathrm{P_F} \leq \mathrm{P_D}$.
  Proof:
  - ROC curve is concave.
  - $(0, 0)$ and $(1, 1)$ belong to the ROC curve.
    - ...thus, $\mathrm{P_F} \leq \mathrm{P_D}$.

Remarks

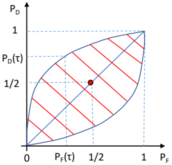


- What about "bad" tests? Are all points with $\mathrm{P_D} \leq \mathrm{P_F}$ feasible? [NO!]
- Assume a "bad" test δ with specific $(\mathrm{P_F}(\delta), \mathrm{P_D}(\delta))$.
- Define the following test $\hat{\delta}$ that flips the decision:

$$\hat{\delta}(\mathbf{y}) = 1 - \delta(\mathbf{y}). \tag{15}$$

- $\hat{\delta}(\mathbf{y})$ achieves $(\widehat{\mathrm{P_F}}, \widehat{\mathrm{P_D}}) = (1 - \mathrm{P_F}(\delta), 1 - \mathrm{P_D}(\delta))$, i.e., region of feasible tests is symmetric around $(1/2, 1/2)$.

Remarks



- Define the following test $\hat{\delta}$ that flips the decision:

$$\hat{\delta}(\mathbf{y}) = 1 - \delta(\mathbf{y}). \tag{16}$$

- $\hat{\delta}(\mathbf{y})$ achieves $(\widehat{\mathrm{P}_\mathrm{F}}, \widehat{\mathrm{P}_\mathrm{D}}) = (1 - \mathrm{P}_\mathrm{F}(\delta), 1 - \mathrm{P}_\mathrm{D}(\delta))$, i.e., region of feasible tests is symmetric around $(1/2, 1/2)$. Proof:

$$\mathrm{P}_\mathrm{F}(\delta) = \int_{\mathcal{Y}_1} f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0) d\mathbf{y} \Rightarrow \tag{17}$$

$$\mathrm{P}_\mathrm{F}(\hat{\delta} = 1 - \delta) = \int_{\mathcal{Y}_0} f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0) d\mathbf{y} = 1 - \mathrm{P}_\mathrm{F}(\delta). \tag{18}$$

...and similarly for $\mathrm{P}_\mathrm{D}(\hat{\delta})$.

Examples

- Assume binary hypothesis testing, with $\mathbf{y} \sim \mathcal{N}(\mathbf{m}_j, \mathbf{K}_j)$ under hypothesis H_j, $j \in \{0, 1\}$ and $\mathbf{y} \in \mathbb{R}^{N \times 1}$.

Preliminaries:

1. For any matrix $\mathbf{K}$ with inverse, $\left(\mathbf{K}^{-1}\right)^T = \left(\mathbf{K}^T\right)^{-1}$.
2. Thus, for any *symmetric* matrix $\mathbf{K}$ with inverse, the inverse is also symmetric: $\left(\mathbf{K}^{-1}\right)^T = \mathbf{K}^{-1}$.
3. For any scalar z with $z = \mathbf{a}^{\mathrm{T}}\mathbf{b}$, $z = \mathbf{b}^{\mathrm{T}}\mathbf{a}$, where $\mathbf{a}, \mathbf{b}$ vectors of the same dimension.

$$f_{\mathbf{y}|\mathrm{H}_j}(\mathbf{y}|\mathrm{H}_j) = \frac{1}{\sqrt{(2\pi)^N |\mathbf{K}_j|}} \exp\left(-\frac{1}{2}(\mathbf{y} - \mathbf{m}_j)^T \mathbf{K}_j^{-1} (\mathbf{y} - \mathbf{m}_j)\right) \Leftrightarrow \tag{19}$$

$$\frac{f_{\mathbf{y}|\mathrm{H}_1}(\mathbf{y}|\mathrm{H}_1)}{f_{\mathbf{y}|\mathrm{H}_0}(\mathbf{y}|\mathrm{H}_0)} \overset{\mathrm{H}_1}{\geq} \tau \Leftrightarrow \tag{20}$$

$$\frac{1}{2}(\mathbf{y} - \mathbf{m}_0)^T \mathbf{K}_0^{-1} (\mathbf{y} - \mathbf{m}_0) - \frac{1}{2}(\mathbf{y} - \mathbf{m}_1)^T \mathbf{K}_1^{-1} (\mathbf{y} - \mathbf{m}_1) + \frac{1}{2}\ln\left(\frac{|\mathbf{K}_0|}{|\mathbf{K}_1|}\right) \overset{\mathrm{H}_1}{\geq} \ln(\tau) \Leftrightarrow \tag{21}$$

...simple calculations exploiting 2. and 3. above [try them!]...

$$\Leftrightarrow \underbrace{\frac{1}{2}\mathbf{y}^T \left(\mathbf{K}_0^{-1} - \mathbf{K}_1^{-1}\right)\mathbf{y} + \mathbf{y}^T \left(\mathbf{K}_1^{-1}\mathbf{m}_1 - \mathbf{K}_0^{-1}\mathbf{m}_0\right)}_{S(\mathbf{y})} \overset{\mathrm{H}_1}{\geq} \eta, \tag{22}$$

$$\eta = \frac{1}{2}\mathbf{m}_1^T \mathbf{K}_1 \mathbf{m}_1 - \frac{1}{2}\mathbf{m}_0^T \mathbf{K}_0 \mathbf{m}_0 - \frac{1}{2}\ln\left(\frac{|\mathbf{K}_0|}{|\mathbf{K}_1|}\right) + \ln(\tau) \tag{23}$$

Examples

- Assume $\mathbf{K}_0 = \mathbf{K}_1 = \mathbf{K}$. In that case, the test is simplified as follows:

$$\mathbf{y}^T\mathbf{K}^{-1}\underbrace{(\mathbf{m}_1 - \mathbf{m}_0)}_{\Delta\mathbf{m}} = \mathbf{y}^T\mathbf{K}^{-1}\Delta\mathbf{m} \overset{\mathrm{H}_1}{\geq} \eta \Leftrightarrow \tag{24}$$

$$S_s(\mathbf{y}) \triangleq \mathbf{y}^T\mathbf{K}^{-1}\Delta\mathbf{m} - \mathbf{m}_0^T\mathbf{K}^{-1}\Delta\mathbf{m} \overset{\mathrm{H}_1}{\geq} \eta - \mathbf{m}_0^T\mathbf{K}^{-1}\Delta\mathbf{m} \triangleq \eta_s \Leftrightarrow \tag{25}$$

$$S_s(\mathbf{y}) = \Delta\mathbf{m}^T\mathbf{K}^{-1}\mathbf{y} - \Delta\mathbf{m}^T\mathbf{K}^{-1}\mathbf{m}_0 \overset{\mathrm{H}_1}{\geq} \eta - \mathbf{m}_0^T\mathbf{K}^{-1}\Delta\mathbf{m} \triangleq \eta_s, \tag{26}$$

where we have used the fact that $\mathbf{K}^{-1}$ is symmetric; $S_s(\mathbf{y})$ is the shifted sufficient statistic, which is affine transformation of a Gaussian vector, and thus, it is also Gaussian:

$$\mathrm{H}_0 : S_s(\mathbf{y}) \sim \mathcal{N}\left(0, \Delta\mathbf{m}^T\mathbf{K}^{-1}\Delta\mathbf{m}\right) \tag{27}$$

$$\mathrm{H}_1 : S_s(\mathbf{y}) \sim \mathcal{N}\left(\Delta\mathbf{m}^T\mathbf{K}^{-1}\Delta\mathbf{m}, \Delta\mathbf{m}^T\mathbf{K}^{-1}\Delta\mathbf{m}\right) \tag{28}$$

Notice that $\mathbf{K}^{-1}$ (and $\mathbf{K}$) are positive definite, and thus, $\Delta\mathbf{m}^T\mathbf{K}^{-1}\Delta\mathbf{m} > 0$. We set $d^2 \triangleq \Delta\mathbf{m}^T\mathbf{K}^{-1}\Delta\mathbf{m}$. We also need the following definition of the (Gauss) Q-function $Q(x)$ and its properties:

$$Q(x) \triangleq \int_x^{+\infty} \frac{1}{2\pi} e^{-t^2/2} dt = 1 - \int_{-\infty}^{x} \frac{1}{2\pi} e^{-t^2/2} dt, \tag{29}$$

$$Q(-x) = 1 - Q(x), \ \ \frac{dQ(x)}{dx} = -\frac{1}{2\pi} e^{-x^2/2}. \tag{30}$$

$$\mathrm{H}_0 : S_s(\mathbf{y}) \sim \mathcal{N}\left(0, \Delta\mathbf{m}^T\mathbf{K}^{-1}\Delta\mathbf{m}\right) \equiv \mathcal{N}\left(0, d^2\right) \tag{31}$$

$$\mathrm{H}_1 : S_s(\mathbf{y}) \sim \mathcal{N}\left(\Delta\mathbf{m}^T\mathbf{K}^{-1}\Delta\mathbf{m}, \Delta\mathbf{m}^T\mathbf{K}^{-1}\Delta\mathbf{m}\right) \equiv \mathcal{N}\left(d^2, d^2\right) \tag{32}$$

We can now calculate the basic probabilities for the test $S_s(\mathbf{y}) \overset{\mathrm{H}_1}{\geq} \eta_s$:

$$\mathrm{P_D} = \int_{\eta_s}^{+\infty} f_{S_s(\mathbf{y})|\mathrm{H}_1}\left(s|\mathrm{H}_1\right)\, ds = \int_{\eta_s}^{+\infty} \frac{1}{\sqrt{2\pi d^2}} \exp\left(-\frac{\left(s-d^2\right)^2}{2d^2}\right) ds \tag{33}$$

$$\overset{\frac{s-d^2}{d}=t}{=} \int_{\frac{\eta_s-d^2}{d}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = Q\left(\frac{\eta_s-d^2}{d}\right) = 1 - Q\left(d - \frac{\eta_s}{d}\right), \tag{34}$$

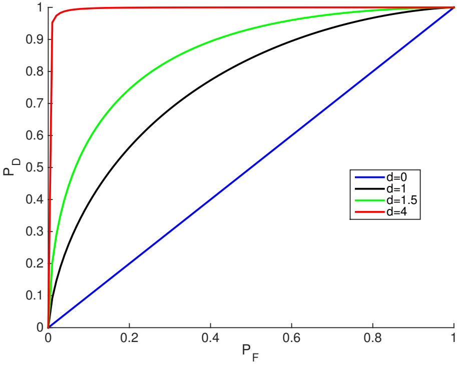
$$\mathrm{P_F} = \int_{\eta_s}^{+\infty} f_{S_s(\mathbf{y})|\mathrm{H}_0}\left(s|\mathrm{H}_0\right)\, ds = \int_{\eta_s}^{+\infty} \frac{1}{\sqrt{2\pi d^2}} \exp\left(-\frac{s^2}{2d^2}\right) ds \tag{35}$$

$$\overset{\frac{s}{d}=t}{=} \int_{\frac{\eta_s}{d}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = Q\left(\frac{\eta_s}{d}\right) \Rightarrow \frac{\eta_s}{d} = Q^{-1}\left(\mathrm{P_F}\right). \tag{36}$$

$$(34), (36) \Rightarrow \mathrm{P_D} = 1 - Q\left(d - Q^{-1}\left(\mathrm{P_F}\right)\right). \tag{37}$$

$$\mathrm{P_D} = 1 - Q\left(d - Q^{-1}\left(\mathrm{P_F}\right)\right). \tag{38}$$

- We need d as large as possible! Why?



Whitening Procedure

- In many cases, it is useful to simplify the observation model with linear transformations. To this end, the eigendecomposition of positive-definite matrix $\mathbf{K}$ is exploited:

$$\mathbf{K}\,\mathbf{P} = \mathbf{P}\boldsymbol{\Lambda} \Leftrightarrow \mathbf{K} \begin{bmatrix} | & | & \cdots & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_N \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \lambda_1 \mathbf{p}_1 & \lambda_2 \mathbf{p}_2 & \cdots & \lambda_N \mathbf{p}_N \\ | & | & \cdots & | \end{bmatrix} \tag{39}$$

$$= \begin{bmatrix} | & | & \cdots & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_N \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix} \Leftrightarrow \mathbf{K} = \mathbf{P}\,\boldsymbol{\Lambda}\,\mathbf{P}^{\mathrm{T}} \tag{40}$$

  with $\mathbf{K}\,\mathbf{p}_i = \lambda_i\,\mathbf{p}_i,\ i \in \{1, 2, \ldots, N\}$, i.e., columns of $\mathbf{P}$ are the eigenvectors of $\mathbf{K}$ and $\{\lambda_i\}$ the corresponding eigenvalues, $\mathbf{P}$ is orthogonal, i.e., $\mathbf{P}\,\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{\mathrm{T}}\,\mathbf{P} = \mathbf{I}_N$ and $\boldsymbol{\Lambda}$ diagonal matrix, with main diagonal the positive eigenvalues of $\mathbf{K}$, i.e., $\boldsymbol{\Lambda} = \mathrm{diag}\,[\lambda_1\,\lambda_2 \ldots \lambda_N]$.

- Set $\boldsymbol{\Lambda}^{-1/2} = \mathrm{diag}\left[\sqrt{\lambda_1}\,\sqrt{\lambda_2}\ldots\sqrt{\lambda_N}\right]$ and multiply $\mathbf{y}$ by $\boldsymbol{\Lambda}^{-1/2}\,\mathbf{P}^{\mathrm{T}}$:

Whitening Procedure

- Multiply $\mathbf{y}$ by $\boldsymbol{\Lambda}^{-1/2}\,\mathbf{P}^{\mathrm{T}}$:

$$\underbrace{\boldsymbol{\Lambda}^{-1/2}\,\mathbf{P}^{\mathrm{T}}\,\mathbf{y}}_{\mathbf{y}_w} = \underbrace{\boldsymbol{\Lambda}^{-1/2}\,\mathbf{P}^{\mathrm{T}}\,\mathbf{m}_j}_{\boldsymbol{\mu}_j} + \underbrace{\boldsymbol{\Lambda}^{-1/2}\,\mathbf{P}^{\mathrm{T}}\,\mathbf{v}}_{\mathbf{v}_w}, \tag{41}$$

$$\Leftrightarrow \mathbf{y}_w = \boldsymbol{\mu}_j + \mathbf{v}_w, \tag{42}$$

with

$$\mathbb{E}[\mathbf{v}_w] = \boldsymbol{\Lambda}^{-1/2}\,\mathbf{P}^{\mathrm{T}}\,\mathbb{E}[\mathbf{v}] = \mathbf{0} \tag{43}$$

$$\mathbb{E}[\mathbf{v}_w\,\mathbf{v}_w^{\mathrm{T}}] = \boldsymbol{\Lambda}^{-1/2}\,\mathbf{P}^{\mathrm{T}}\,\mathbf{P}\,\boldsymbol{\Lambda}\,\mathbf{P}^{\mathrm{T}}\,\mathbf{P}\,\boldsymbol{\Lambda}^{-1/2} = \boldsymbol{\Lambda}^{-1/2}\boldsymbol{\Lambda}\,\boldsymbol{\Lambda}^{-1/2} = \mathbf{I}_N. \tag{44}$$

- Thus, $\mathbf{v}_w \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_N)$ and the detection problem is simplified in Eq. (42). Notice that

$$\delta\boldsymbol{\mu} \stackrel{\triangle}{=} \boldsymbol{\mu}_1 - \boldsymbol{\mu}_0 = \boldsymbol{\Lambda}^{-1/2}\,\mathbf{P}^{\mathrm{T}}\,(\mathbf{m}_1 - \mathbf{m}_0) = \boldsymbol{\Lambda}^{-1/2}\,\mathbf{P}^{\mathrm{T}}\,\delta\mathbf{m} \tag{45}$$

$$\begin{aligned} d^2 &\stackrel{\triangle}{=} \delta\mathbf{m}^{\mathrm{T}}\mathbf{K}^{-1}\delta\mathbf{m} = \delta\mathbf{m}^{\mathrm{T}}\mathbf{P}\,\boldsymbol{\Lambda}^{-1}\,\mathbf{P}^{\mathrm{T}}\delta\mathbf{m} = \delta\mathbf{m}^{\mathrm{T}}\mathbf{P}\,\boldsymbol{\Lambda}^{-1/2}\,\boldsymbol{\Lambda}^{-1/2}\,\mathbf{P}^{\mathrm{T}}\delta\mathbf{m} \equiv \delta\boldsymbol{\mu}^{\mathrm{T}}\,\delta\boldsymbol{\mu} \\ &= \|\delta\boldsymbol{\mu}\|_2^2. \end{aligned} \tag{46}$$

Examples

- Another example:

$$\mathbf{y} = \boldsymbol{\mu}_j + \mathbf{v}, \tag{47}$$

with $\mathbf{v} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2 \mathbf{I}_N\right)$, $j \in \{0, 1\}$ and $\pi_0 = \pi_1 = 1/2$.
It can be easily shown that:

1. Minimum probability of error detection rule is the minimum distance rule:
$$\|\mathbf{y} - \boldsymbol{\mu}_0\|_2 \overset{H_1}{\geq} \|\mathbf{y} - \boldsymbol{\mu}_1\|_2$$
2. The probability of error of the above rule is given by:
$$\Pr(e) = Q\left(\frac{\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0\|_2}{2\sigma}\right).$$

Proof: simply write the ML rule and exploit the fact that affine transformation of Gaussian vectors is also Gaussian.

References

[1] Bernard C. Levy, Principles of Signal Detection and Parameter Estimation, Springer 2008.
[2] Instructor notes.

Thank you!

Detection & Estimation Theory: Lectures 7 & 8

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Outline

- M-ary Hypothesis Testing
- Error Probability Bounds
- Example

M-ary Hypothesis Testing

- Assume M hypotheses $\{\mathrm{H}_j\}$, $j \in \{0,\ 1, \ldots, M-1\}$, with priors $\Pr(\mathrm{H}_j) \triangleq \pi_j$.
  1. Observe $\mathbf{y}$ and find out decision rule $\delta(\mathbf{y}) = j$, i.e., decide H_j for that specific $\mathbf{y}$.
  2. Equivalently, find out $\mathcal{Y}_i = \{\mathbf{y} :\ \delta(\mathbf{y}) = i\}$, with $\mathcal{Y} = \cup_{i=0}^{M-1} \mathcal{Y}_i$ and $\mathcal{Y}_i \cap \mathcal{Y}_j = \emptyset \, \forall i \neq j$.
- We will revert to Bayesian formulation. Remember Bayes Risk $R(\delta)$ and conditional Bayes Risk $R(\delta|\mathrm{H}_j)$:

$$R(\delta) = \sum_{j=0}^{M-1} R(\delta|\mathrm{H}_j)\pi_j \tag{1}$$

$$R(\delta|\mathrm{H}_j) = \sum_{i=0}^{M-1} C_{ij} \Pr(\delta(\mathbf{y}) = i|\mathrm{H}_j) = \sum_{i=0}^{M-1} C_{ij} \Pr(\mathcal{Y}_i|\mathrm{H}_j)$$

$$\Leftrightarrow R(\delta) = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} C_{ij} \Pr(\mathcal{Y}_i|\mathrm{H}_j)\, \pi_j, \tag{2}$$

where C_{ij} is the cost of deciding i when H_j holds.

M-ary Hypothesis Testing

- Bayesian formulation:

$$R(\delta) = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} C_{ij} \, \Pr\left(\mathcal{Y}_i | \mathrm{H}_j\right) \pi_j \tag{3}$$

$$= \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} C_{ij} \int_{\mathcal{Y}_i} f_{\mathbf{y}|\mathrm{H}_j}\left(\mathbf{y}|\mathrm{H}_j\right) \pi_j \, d\mathbf{y} \tag{4}$$

$$= \sum_{i=0}^{M-1} \int_{\mathcal{Y}_i} \sum_{j=0}^{M-1} C_{ij} \, f_{\mathbf{y}|\mathrm{H}_j}\left(\mathbf{y}|\mathrm{H}_j\right) \pi_j \, d\mathbf{y} \tag{5}$$

$$= \sum_{i=0}^{M-1} \int_{\mathcal{Y}_i} \sum_{j=0}^{M-1} C_{ij} \, \Pr\left(\mathrm{H}_j|\mathbf{y}\right) f_{\mathbf{y}}\left(\mathbf{y}\right) d\mathbf{y} \tag{6}$$

$$= \sum_{i=0}^{M-1} \int_{\mathcal{Y}_i} f_{\mathbf{y}}\left(\mathbf{y}\right) \underbrace{\sum_{j=0}^{M-1} C_{ij} \, \Pr\left(\mathrm{H}_j|\mathbf{y}\right)}_{C_i(\mathbf{y})} d\mathbf{y} \tag{7}$$

M-ary Hypothesis Testing

- Bayesian formulation:

$$R(\delta) = \sum_{i=0}^{M-1} \int_{\mathcal{Y}_i} f_{\mathbf{y}}(\mathbf{y}) \underbrace{\sum_{j=0}^{M-1} C_{ij} \Pr(\mathrm{H}_j | \mathbf{y})}_{C_i(\mathbf{y})} \, d\mathbf{y} \tag{8}$$

$$= \sum_{i=0}^{M-1} \int_{\mathcal{Y}_i} C_i(\mathbf{y}) \, f_{\mathbf{y}}(\mathbf{y}) \, d\mathbf{y}. \tag{9}$$

- From Eq. (9),

$$\delta_B(\mathbf{y}) = \arg \min_{i \in \{0,1,\ldots,M-1\}} C_i(\mathbf{y}) \tag{10}$$

i.e., we select H_k if $C_k(\mathbf{y}) \leq C_i(\mathbf{y})$, $\forall i \in \{0, 1, \ldots, M-1\}$.

min Pr(e) rule: MAP rule

- Set symmetric costs:

$$C_{ij} = 1 - \delta_{ij} = \begin{cases} 0, & i = j, \\ 1, & i \neq j. \end{cases} \tag{11}$$

In that case, min of prob. of error is equivalent to risk minimization (as in the binary case):

$$R(\delta|\mathrm{H}_j) = \sum_{i=0}^{M-1} C_{ij} \Pr\left(\mathcal{Y}_i|\mathrm{H}_j\right) = \sum_{i \neq j} \Pr\left(\mathcal{Y}_i|\mathrm{H}_j\right) \tag{12}$$

$$\Rightarrow R(\delta|\mathrm{H}_j) = 1 - \Pr\left(\mathcal{Y}_j|\mathrm{H}_j\right) \equiv \Pr\left(e|\mathrm{H}_j\right) \tag{13}$$

$$R(\delta) = \sum_{j=0}^{M-1} R(\delta|\mathrm{H}_j)\pi_j = \sum_{j=0}^{M-1} \Pr\left(e|\mathrm{H}_j\right)\pi_j \equiv \Pr(e) \tag{14}$$

$$C_i(\mathbf{y}) = \sum_{j=0}^{M-1} C_{ij} \Pr\left(\mathrm{H}_j|\mathbf{y}\right) = \sum_{j \neq i} \Pr\left(\mathrm{H}_j|\mathbf{y}\right) = 1 - \Pr\left(\mathrm{H}_i|\mathbf{y}\right) \tag{15}$$

min $\Pr(e)$ rule: MAP rule

- Set symmetric costs:

$$C_{ij} = 1 - \delta_{ij} = \begin{cases} 0, & i = j, \\ 1, & i \neq j. \end{cases} \tag{16}$$

- Min prob. of error rule:

$$C_i(\mathbf{y}) = \sum_{j=0}^{M-1} C_{ij} \Pr\left(\mathrm{H}_j|\mathbf{y}\right) = \sum_{j \neq i} \Pr\left(\mathrm{H}_j|\mathbf{y}\right) = 1 - \Pr\left(\mathrm{H}_i|\mathbf{y}\right)$$

$$\delta_{MAP}(\mathbf{y}) = \arg \min_{i \in \{0,1,\ldots,M-1\}} \left\{1 - \Pr\left(\mathrm{H}_i|\mathbf{y}\right)\right\} \tag{17}$$

$$= \arg \max_{i \in \{0,1,\ldots,M-1\}} \Pr\left(\mathrm{H}_i|\mathbf{y}\right), \text{ (MAP rule)} \tag{18}$$

$$= \arg \max_{i \in \{0,1,\ldots,M-1\}} \frac{f_{\mathbf{y}|\mathrm{H}_i}\left(\mathbf{y}|\mathrm{H}_i\right) \pi_i}{f_{\mathbf{y}}(\mathbf{y})} \tag{19}$$

$$= \arg \max_{i \in \{0,1,\ldots,M-1\}} f_{\mathbf{y}|\mathrm{H}_i}\left(\mathbf{y}|\mathrm{H}_i\right) \pi_i \tag{20}$$

min Pr(e) rule and equiprobable hypotheses: ML rule

- Set symmetric costs and equiprobable hypotheses ($\pi_j = 1/M$):

$$C_{ij} = 1 - \delta_{ij} = \begin{cases} 0, & i = j, \\ 1, & i \neq j. \end{cases} \tag{21}$$

- Min prob. of error rule:

$$C_i(\mathbf{y}) = \sum_{j=0}^{M-1} C_{ij} \, \Pr\left(\mathrm{H}_j | \mathbf{y}\right) = \sum_{j \neq i} \Pr\left(\mathrm{H}_j | \mathbf{y}\right) = 1 - \Pr\left(\mathrm{H}_i | \mathbf{y}\right)$$

$$\delta_{ML}(\mathbf{y}) = \arg \max_{i \in \{0,1,\ldots,M-1\}} f_{\mathbf{y}|\mathrm{H}_i}\left(\mathbf{y}|\mathrm{H}_i\right) \cdot (\pi_i = 1/M) \tag{22}$$

$$= \arg \max_{i \in \{0,1,\ldots,M-1\}} f_{\mathbf{y}|\mathrm{H}_i}\left(\mathbf{y}|\mathrm{H}_i\right) \text{ (ML rule).} \tag{23}$$

- ...generalisation of the binary hypothesis testing case!

Example

- Under hypohesis H_i, with $i \in \{0, 1, \dots, M-1\}$, $\pi_i = 1/M$ and $\mathbf{y} = [y_1 \ y_2 \ \dots \ y_N]^{\mathrm{T}}$:

$$\mathbf{y} \sim \mathcal{N}(\mathbf{m}_i, \mathbf{K}) \,. \tag{24}$$

  What is the minimum probability of error detection rule?

- Minimum probability of error detection rule for equiprobable hypotheses is the ML rule:

$$\delta_{ML}(\mathbf{y}) = \arg \max_{i \in \{0,1,\dots,M-1\}} f_{\mathbf{y}|\mathrm{H}_i}(\mathbf{y}|\mathrm{H}_i) \tag{25}$$

$$= \arg \max_{i \in \{0,1,\dots,M-1\}} \ln \left[f_{\mathbf{y}|\mathrm{H}_i}(\mathbf{y}|\mathrm{H}_i) \right] \tag{26}$$

$$= \arg \max_{i \in \{0,1,\dots,M-1\}} \ln \left[-\frac{1}{2} \left((\mathbf{y} - \mathbf{m}_i)^{\mathrm{T}} \mathbf{K}^{-1} (\mathbf{y} - \mathbf{m}_i) \right) - \frac{N}{2} \ln(2\pi) - \frac{\ln(|\mathbf{K}|)}{2} \right] \tag{27}$$

$$= \arg \min_{i \in \{0,1,\dots,M-1\}} \ln \left[\left((\mathbf{y} - \mathbf{m}_i)^{\mathrm{T}} \mathbf{K}^{-1} (\mathbf{y} - \mathbf{m}_i) \right) \right] \tag{28}$$

$$= \arg \min_{i \in \{0,1,\dots,M-1\}} \left((\mathbf{y} - \mathbf{m}_i)^{\mathrm{T}} \mathbf{K}^{-1} (\mathbf{y} - \mathbf{m}_i) \right) \tag{29}$$

- Note that through whitening $\mathbf{z} = \mathbf{\Lambda}^{-1/2} \mathbf{P}^{\mathrm{T}} \mathbf{y}$, i.e., using $\mathbf{K} = \mathbf{P} \mathbf{\Lambda} \mathbf{P}^{\mathrm{T}}$, under hypothesis H_i:

$$\mathbf{z} \sim \mathcal{N}\left(\mathbf{\Lambda}^{-1/2} \mathbf{P}^{\mathrm{T}} \mathbf{m}_i, \mathbf{I}_N \right) . \tag{30}$$

  ...analytical proof in the previous lectures.

Error Probability Bounds

- Set symmetric costs:

$$C_{ij} = 1 - \delta_{ij} = \begin{cases} 0, & i = j, \\ 1, & i \neq j. \end{cases} \tag{31}$$

As we have already seen:

$$R(\delta|\mathrm{H}_j) = \sum_{i \neq j} \Pr\left(\mathcal{Y}_i|\mathrm{H}_j\right) = 1 - \Pr\left(\mathcal{Y}_j|\mathrm{H}_j\right) \equiv \Pr\left(e|\mathrm{H}_j\right) \triangleq \Pr\left(\mathcal{Y}_j^c|\mathrm{H}_j\right) \tag{32}$$

$$R(\delta) = \sum_{j=0}^{M-1} R(\delta|\mathrm{H}_j)\pi_j = \sum_{j=0}^{M-1} \Pr\left(e|\mathrm{H}_j\right)\pi_j \equiv \Pr(e) \tag{33}$$

- So, $\mathcal{Y}_j^c$ is the region where hypothesis H_j is NOT selected. Define formally the following:

$$\mathcal{Y}_j^c = \bigcup_{k \neq j} \mathcal{E}_{kj}, \tag{34}$$

$$\mathcal{E}_{kj} = \left\{\mathbf{y} : \Pr\left(\mathrm{H}_k|\mathbf{y}\right) > \Pr\left(\mathrm{H}_j|\mathbf{y}\right)\right\} \tag{35}$$

$$= \left\{\mathbf{y} : \frac{f_{\mathbf{y}|\mathrm{H}_k}\left(\mathbf{y}|\mathrm{H}_k\right)}{f_{\mathbf{y}|\mathrm{H}_j}\left(\mathbf{y}|\mathrm{H}_j\right)} > \frac{\pi_j}{\pi_k}\right\}, \tag{36}$$

i.e., $\mathcal{E}_{kj}$ is the region of $\{\mathbf{y}\}$'s where hypothesis H_k is preferred over H_j.

Error Probability Bounds

- The areas $\{\mathcal{E}_{kj}\}$ for given j usually overlap.
- It is often possible to find a subset of such areas, such that:

$$\mathcal{Y}_j^c = \bigcup_{k \in N(j)} \mathcal{E}_{kj}, \text{ with } N(j) \subset \{0,\ 1,\ \ldots,\ M-1\} \,/\, j \quad (37)$$

  i.e., $N(j)$ does not include element j.

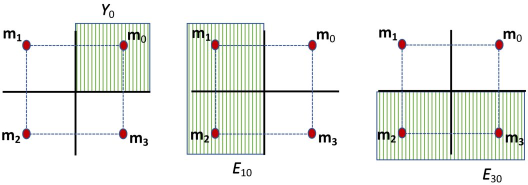
- Thus,

$$\Pr(e|\mathrm{H}_j) \triangleq \Pr\left(\mathcal{Y}_j^c|\mathrm{H}_j\right) \leq \underbrace{\sum_{k \in N(j)} \Pr(\mathcal{E}_{kj}|\mathrm{H}_j)}_{\text{Improved union bound}} \quad (38)$$

$$\max_{k \neq j} \Pr(\mathcal{E}_{kj}|\mathrm{H}_j) \leq \Pr\left(\mathcal{Y}_j^c|\mathrm{H}_j\right) \quad (39)$$

- The above bounds are usually simple to calculate!

Example



- Find error probability for optimal detection for the following:

$$\mathrm{H}_j : \ \mathbf{y} = \boldsymbol{m}_j + \mathbf{v}, \tag{40}$$

with $\mathbf{v} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2 \mathbf{I}_2\right)$, $j \in \{0, 1, 2, 3\}$ and $\pi_j = 1/4$.

- MAP is simplified to ML, simplified to minimum distance. In addition:

$$\mathcal{Y}_0^c = \mathcal{E}_{10} \cup \mathcal{E}_{30} \tag{41}$$

$$d \triangleq \|\boldsymbol{m}_1 - \boldsymbol{m}_0\|_2 = \|\boldsymbol{m}_2 - \boldsymbol{m}_1\|_2 = \|\boldsymbol{m}_3 - \boldsymbol{m}_2\|_2 = \|\boldsymbol{m}_3 - \boldsymbol{m}_0\|_2 \tag{42}$$

$$\Pr(e|\mathrm{H}_0) = \Pr\left(\mathcal{Y}_0^c|\mathrm{H}_0\right) = \Pr(\mathcal{E}_{10} \cup \mathcal{E}_{30}|\mathrm{H}_0) \leq \underbrace{\Pr(\mathcal{E}_{10}|\mathrm{H}_0)}_{Q\left(\frac{d}{2\sigma}\right)} + \underbrace{\Pr(\mathcal{E}_{30}|\mathrm{H}_0)}_{Q\left(\frac{d}{2\sigma}\right)} \tag{43}$$

$$\Leftrightarrow \Pr(e|\mathrm{H}_0) \leq 2Q\left(\frac{d}{2\sigma}\right), \tag{44}$$

where the latter is due to minimum distance binary error detection in white Gaussian noise (reminder in next slide).

Reminder

- Reminder:

$$\mathrm{H}_j: \ \mathbf{y} = \boldsymbol{\mu}_j + \mathbf{v}, \tag{45}$$

with $\mathbf{v} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2 \mathbf{I}_N\right)$, $j \in \{0, 1\}$ and $\pi_0 = \pi_1 = 1/2$.

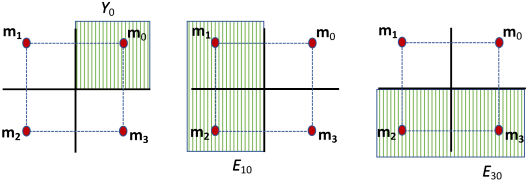
1. Minimum probability of error detection rule is the minimum distance rule:

$$\|\mathbf{y} - \boldsymbol{\mu}_0\|_2 \overset{H_1}{\geq} \|\mathbf{y} - \boldsymbol{\mu}_1\|_2$$

2. The probability of error of the above rule is given by:

$$\Pr(e) = Q\left(\frac{\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0\|_2}{2\sigma}\right).$$

Example



- Error analysis:

$$\mathcal{Y}_0^c = \mathcal{E}_{10} \cup \mathcal{E}_{30} \tag{46}$$

$$d \triangleq \|\boldsymbol{m}_1 - \boldsymbol{m}_0\|_2 = \|\boldsymbol{m}_2 - \boldsymbol{m}_1\|_2 = \|\boldsymbol{m}_3 - \boldsymbol{m}_2\|_2 = \|\boldsymbol{m}_3 - \boldsymbol{m}_0\|_2 \tag{47}$$

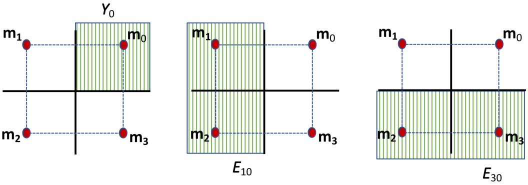
$$\Pr(e|\mathrm{H}_0) = \Pr\left(\mathcal{Y}_0^c|\mathrm{H}_0\right) = \Pr(\mathcal{E}_{10} \cup \mathcal{E}_{30}|\mathrm{H}_0) \le \underbrace{\Pr(\mathcal{E}_{10}|\mathrm{H}_0)}_{Q\left(\frac{d}{2\sigma}\right)} + \underbrace{\Pr(\mathcal{E}_{30}|\mathrm{H}_0)}_{Q\left(\frac{d}{2\sigma}\right)} = 2Q\left(\frac{d}{2\sigma}\right)$$

$$Q\left(\frac{d}{2\sigma}\right) = \Pr(\mathcal{E}_{10}|\mathrm{H}_0) = \Pr(\mathcal{E}_{30}|\mathrm{H}_0) \le \Pr(e|\mathrm{H}_0) \tag{48}$$

$$\Pr(e|\mathrm{H}_0) = \Pr(e)\,, \text{ due to symmetry} \tag{49}$$

$$\Rightarrow Q\left(\frac{d}{2\sigma}\right) \le \Pr(e) \le 2Q\left(\frac{d}{2\sigma}\right), \tag{50}$$

Example



- Error analysis with improved union bound:

$$d \triangleq \|\boldsymbol{m}_1 - \boldsymbol{m}_0\|_2 = \|\boldsymbol{m}_2 - \boldsymbol{m}_1\|_2 = \|\boldsymbol{m}_3 - \boldsymbol{m}_2\|_2 = \|\boldsymbol{m}_3 - \boldsymbol{m}_0\|_2 \tag{51}$$

$$Q\left(\frac{d}{2\sigma}\right) \leq \Pr(e) \leq 2Q\left(\frac{d}{2\sigma}\right) \tag{52}$$

- Exact error analysis:

$$\Pr(e) = 1 - \left(1 - Q\left(\frac{d}{2\sigma}\right)\right)^2 \tag{53}$$

$$= 2Q\left(\frac{d}{2\sigma}\right) - \left[Q\left(\frac{d}{2\sigma}\right)\right]^2 \tag{54}$$

Thus, upper error probability bound is tight!

References

[1] Bernard C. Levy, Principles of Signal Detection and Parameter Estimation, Springer 2008.
[2] Instructor notes.

Thank you!

Detection & Estimation Theory: Lectures 9-10

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Technical University of Crete
School of Electrical and Computer Engineering

European Union
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Human Resources Development,
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ΕΣΠΑ
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Outline

- Bayesian Estimation: Problem Definition
    - Details on Problem Formulation
- Optimum Bayesian Estimator
    - Bayesian MSE Estimator
        MSE Performance Evaluation
    - Bayesian MAE Estimator
    - Bayesian MAP Estimator

Parameter Estimation Theory: Bayesian Formulation

- Formulation for Bayesian estimation is similar to Bayesian detection!
  1. Observe $\mathbf{y} \in \mathbb{R}^n$ for estimation of parameter vector $\mathbf{x} \in \mathbb{R}^m$.
  2. Detection: find out decision rule $\delta(\mathbf{y}) = j$, where j is discrete.
  3. Estimation: find out estimate $\hat{\mathbf{x}}(\mathbf{y}) \in \mathbb{R}^m$.
- Formulation for Bayesian estimation requires the following:

  1. Observation model: $f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})$, i.e., conditional p.d.f. from measurements!
  2. Prior density: $f_{\mathbf{x}}(\mathbf{x})$, i.e., prior p.d.f. density of the unknown parameter. Notice that in the Bayesian formulation the unknown parameter is assumed random!
  3. Cost function: $C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x}) \equiv C(\hat{\mathbf{x}}, \mathbf{x})$, i.e., the cost of estimating $\mathbf{x}$ as $\hat{\mathbf{x}}(\mathbf{y})$.
- Derivations in Bayesian estimation proceed alongside similar lines to Bayesian detection!
  ...some details on the problem formulation follow...

Details on problem formulation

- Observation model $f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})$ is defined explicitly or indirectly through a measurement model.
- One example: $\mathbf{y} = \mathbf{h}(\mathbf{x}) + \mathbf{v}$, with $f_{\mathbf{v}}(\mathbf{v})$ known and $\mathbf{h}(\mathbf{x})$ is a deterministic vector function of $\mathbf{x}$; assume $n = m$.
  $\Rightarrow f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \overset{n=m}{=} f_{\mathbf{v}}(\mathbf{y} - \mathbf{h}(\mathbf{x}))$.

Proof:

$$\mathbf{J}_{n\times m}(\mathbf{x},\mathbf{y}) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_m} \end{bmatrix} = \begin{bmatrix} \nabla y_1 \\ \nabla y_2 \\ \vdots \\ \nabla y_n \end{bmatrix}, \tag{1}$$

$$\text{where } \nabla f = [\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \cdots \quad \frac{\partial f}{\partial x_m}], \tag{2}$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) + \mathbf{v} \Rightarrow$$

$$f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \overset{n=m}{=} \frac{f_{\mathbf{v}}(\mathbf{v})}{\det(\mathbf{J}(\mathbf{x},\mathbf{y}))}\Big|_{\mathbf{v}=\mathbf{y}-\mathbf{h}(\mathbf{x})} = \frac{f_{\mathbf{v}}(\mathbf{y}-\mathbf{h}(\mathbf{x}))}{\det(\mathbf{I}_n)} = f_{\mathbf{v}}(\mathbf{y}-\mathbf{h}(\mathbf{x})) \tag{3}$$

Details on problem formulation

- Prior density $f_{\mathbf{x}}(\mathbf{x})$ is known.
- ...unfortunately, prior density biases the estimator towards more probable values of $\mathbf{x}$, i.e., values of $\mathbf{x}$, where $f_{\mathbf{x}}(\mathbf{x})$ is larger.
- ...remember that we don't know anything about $f_{\mathbf{x}}(\mathbf{x})$ (i.e., it is random), apart from possible values.

Details on problem formulation

- Cost function: $C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x}) \equiv C(\hat{\mathbf{x}}, \mathbf{x})$, i.e., the cost of estimating $\mathbf{x}$ as $\hat{\mathbf{x}}(\mathbf{y})$.

$$C(\hat{\mathbf{x}}, \mathbf{x}) = C(\hat{\mathbf{x}} - \mathbf{x}) \equiv L(\mathbf{e}) \tag{4}$$

- Loss function $L(\mathbf{e})$ is a not decreasing function of error $\mathbf{e} \stackrel{\triangle}{=} \hat{\mathbf{x}}(\mathbf{y}) - \mathbf{x}$.
- 3 different versions of loss function are typically used:
  1. $L_{\text{MSE}}(\mathbf{e}) = ||\mathbf{e}||_2^2$
  2. $L_{\text{MAE}}(\mathbf{e}) = ||\mathbf{e}||_1$
  3. $L_\epsilon(\mathbf{e})$: notch function.

Details on problem formulation

- 3 different versions of loss function are typically used:
  1. Euclidean norm or 2-norm squared:

     $$L_{\mathrm{MSE}}(\mathbf{e}) = ||\mathbf{e}||_2^2 = \mathbf{e}^{\mathrm{T}}\,\mathbf{e} = \sum_{i=1}^{m} e_i^2. \tag{5}$$

     Due to the square, it penalises more larger errors; improbable instances of $\mathbf{x}$ matter a lot; sensitive to modelling errors.
  2. Sum norm or 1-norm:

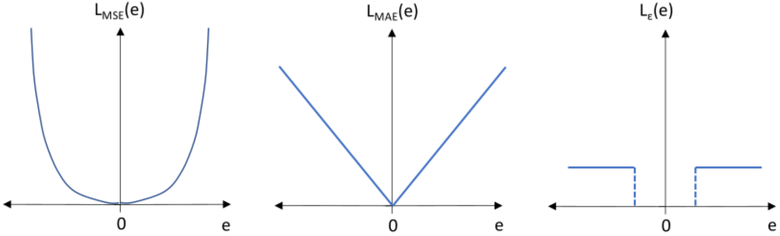
     $$L_{\mathrm{MAE}}(\mathbf{e}) = ||\mathbf{e}||_1 = \sum_{i=1}^{m} |e_i|. \tag{6}$$

     It weights equally the magnitude of all errors.
  3. using the infinity norm:

     $$L_{\epsilon}(\mathbf{e}) = \begin{cases} 0, & \text{if } ||\mathbf{e}||_\infty < \epsilon \\ 1, & \text{otherwise,} \end{cases} \tag{7}$$

     where infinity norm is given by $||\mathbf{e}||_\infty = \max_i |e_i|$; notch function cares about small errors and not about errors above ϵ.

Details on problem formulation: error (cost) functions



- The first two are convex, while the last one is non-convex (for scalar error).

Optimum Bayesian Estimator

- Cost is a function of random vectors, since both $\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x}$ are random.
- Bayesian objective: find $\hat{\mathbf{x}}(\mathbf{y}) = [x_1(\mathbf{y}) \;\; x_2(\mathbf{y}) \;\; \ldots \;\; x_m(\mathbf{y})]^{\mathrm{T}}$ by minimising the expected cost:

$$\min \mathbb{E}\left[C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x})\right], \text{ where}$$

$$\mathbb{E}\left[C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x})\right] = \int_{\mathbf{x}} \int_{\mathbf{y}} C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x}) \; f_{\mathbf{x},\mathbf{y}}(\mathbf{x}, \mathbf{y}) \; d\mathbf{x} \; d\mathbf{y} \tag{8}$$

$$= \int_{\mathbf{y}} \left[\int_{\mathbf{x}} C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x}) \; f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) \; d\mathbf{x} \right] \; f_{\mathbf{y}}(\mathbf{y}) \; d\mathbf{y}. \tag{9}$$

- Notice that posterior p.d.f. $f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})$ and measurements p.d.f. $f_{\mathbf{y}}(\mathbf{y})$ can be (at least in principle) known:

$$f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) = \frac{f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \; f_{\mathbf{x}}(\mathbf{x})}{f_{\mathbf{y}}(\mathbf{y})}, \; f_{\mathbf{y}}(\mathbf{y}) = \int_{\mathbf{x}} f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \; f_{\mathbf{x}}(\mathbf{x}) \; d\mathbf{x} \tag{10}$$

Optimum Bayesian Estimator

$$\min \mathbb{E}\left[C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x})\right], \text{ where}$$

$$\mathbb{E}\left[C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x})\right] = \int_{\mathbf{y}} \left[\int_{\mathbf{x}} C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x})\ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x}\right]\ f_{\mathbf{y}}(\mathbf{y})\ d\mathbf{y}. \tag{11}$$

- Notice that since measurements p.d.f. $f_{\mathbf{y}}(\mathbf{y})$ is non-negative for each given $\mathbf{y}$, the term between brackets above is minimised for each given $\mathbf{y}$ according to the following:

$$\begin{aligned}
\hat{\mathbf{x}}(\mathbf{y}) &= \arg\min_{\mathbf{x}} \int_{\mathbf{x}} C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x})\ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} && (12)\\
&= \arg\min_{\mathbf{x}} \frac{1}{f_{\mathbf{y}}(\mathbf{y})} \int_{\mathbf{x}} C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x})\ f_{\mathbf{x},\mathbf{y}}(\mathbf{x}, \mathbf{y})\ d\mathbf{x} && (13)\\
&= \arg\min_{\mathbf{x}} \int_{\mathbf{x}} C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x})\ f_{\mathbf{x},\mathbf{y}}(\mathbf{x}, \mathbf{y})\ d\mathbf{x} && (14)\\
&= \arg\min_{\mathbf{x}} \int_{\mathbf{x}} C(\hat{\mathbf{x}}(\mathbf{y}), \mathbf{x})\ f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\ f_{\mathbf{x}}(\mathbf{x})\ d\mathbf{x}. && (15)
\end{aligned}$$

Optimum Bayesian Estimator: MSE case

- For minimum square error (MSE) loss function:

$$C(\hat{\mathbf{x}}(\mathbf{y}),\mathbf{x}) = (\underbrace{\hat{\mathbf{x}}-\mathbf{x}}_{\mathbf{e}})^{\mathrm{T}}(\hat{\mathbf{x}}-\mathbf{x}) = L_{\mathrm{MSE}}(\mathbf{e}) = ||\mathbf{e}||_2^2 \tag{16}$$

$$= \hat{\mathbf{x}}^{\mathrm{T}}\hat{\mathbf{x}} - \hat{\mathbf{x}}^{\mathrm{T}}\mathbf{x} - \mathbf{x}^{\mathrm{T}}\hat{\mathbf{x}} + \mathbf{x}^{\mathrm{T}}\mathbf{x} \tag{17}$$

- Since $\mathbf{a}^{\mathrm{T}}\mathbf{b} = \mathbf{b}^{\mathrm{T}}\mathbf{a}$, $\frac{\partial \mathbf{b}^{\mathrm{T}}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{b}$ and $\frac{\partial \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = \left(\mathbf{A}+\mathbf{A}^{\mathrm{T}}\right)\mathbf{x}$,

$$\frac{\partial C(\hat{\mathbf{x}}(\mathbf{y}),\mathbf{x})}{\partial \hat{\mathbf{x}}} = 2\left(\mathbf{I}+\mathbf{I}^{\mathrm{T}}\right)\hat{\mathbf{x}} - \mathbf{x} - \mathbf{x} = 2(\hat{\mathbf{x}}-\mathbf{x}) \tag{18}$$

- Denoting $\nabla_{\hat{\mathbf{x}}} = \left[\frac{\partial}{\partial \hat{x}_1} \quad \frac{\partial}{\partial \hat{x}_2} \quad \ldots \quad \frac{\partial}{\partial \hat{x}_m}\right]^{\mathrm{T}} = \frac{\partial}{\partial \hat{\mathbf{x}}}$, and the following (scalar) function of $\hat{\mathbf{x}}$ (from Eq. (12)),

$$J(\hat{\mathbf{x}}|\mathbf{y}) = \int_{\mathbf{x}} C(\hat{\mathbf{x}}(\mathbf{y}),\mathbf{x})\ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} \tag{19}$$

$$\Rightarrow \nabla_{\hat{\mathbf{x}}} J(\hat{\mathbf{x}}|\mathbf{y}) = \int_{\mathbf{x}} \nabla_{\hat{\mathbf{x}}} C(\hat{\mathbf{x}}(\mathbf{y}),\mathbf{x})\ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} = \mathbf{0} \tag{20}$$

$$= \int_{\mathbf{x}} 2(\hat{\mathbf{x}}-\mathbf{x})\ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} = \mathbf{0} \tag{21}$$

Optimum Bayesian Estimator: MSE case

- ...continued from previous page...

$$\int_{\mathbf{x}} 2(\hat{\mathbf{x}} - \mathbf{x})\ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} = \mathbf{0} \tag{22}$$

$$\Leftrightarrow \int_{\mathbf{x}} \hat{\mathbf{x}}\ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} = \int_{\mathbf{x}} \mathbf{x}\ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} \tag{23}$$

$$\Leftrightarrow \hat{\mathbf{x}} \int_{\mathbf{x}} \ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} = \int_{\mathbf{x}} \mathbf{x}\ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} \stackrel{\triangle}{=} \mathbb{E}\left[\mathbf{x}|\mathbf{y}\right] \tag{24}$$

$$\Leftrightarrow \hat{\mathbf{x}}(\mathbf{y})_{\mathrm{MSE}} = \int_{\mathbf{x}} \mathbf{x}\ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} \stackrel{\triangle}{=} \mathbb{E}\left[\mathbf{x}|\mathbf{y}\right]. \tag{25}$$

- Bayesian MSE estimator is the conditional (on $\mathbf{y}$) mean!

MSE Performance Evaluation

- First, conditional mean square error is calculated:

$$
\begin{aligned}
J(\hat{\mathbf{x}}(\mathbf{y})|\mathbf{y}) &\overset{\hat{\mathbf{x}}(\mathbf{y})\equiv\hat{\mathbf{x}}}{=} \int_{\mathbf{x}} (\hat{\mathbf{x}}-\mathbf{x})^{\mathrm{T}}(\hat{\mathbf{x}}-\mathbf{x})\ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} && (26)\\
&\overset{\triangle}{=} \mathbb{E}_{\mathbf{x}|Y=\mathbf{y}}\left[(\hat{\mathbf{x}}-\mathbf{x})^{\mathrm{T}}(\hat{\mathbf{x}}-\mathbf{x})|Y=\mathbf{y}\right] && (27)\\
&\overset{(*)}{=} \mathbb{E}_{\mathbf{x}|Y=\mathbf{y}}\left[\mathrm{Trace}\left\{(\hat{\mathbf{x}}-\mathbf{x})^{\mathrm{T}}(\hat{\mathbf{x}}-\mathbf{x})\right\}|Y=\mathbf{y}\right] && (28)\\
&\overset{(**)}{=} \mathrm{Trace}\left\{\mathbb{E}_{\mathbf{x}|Y=\mathbf{y}}\left[(\hat{\mathbf{x}}-\mathbf{x})(\hat{\mathbf{x}}-\mathbf{x})^{\mathrm{T}}|Y=\mathbf{y}\right]\right\} && (29)\\
&= \mathrm{Trace}\left\{\mathbf{K}_{\mathbf{x}|Y=\mathbf{y}}\right\}, && (30)
\end{aligned}
$$

where $\mathrm{Trace}(\mathbf{A}\,\mathbf{B}) = \mathrm{Trace}(\mathbf{B}\,\mathbf{A})$ property was used in $(*)$ and $\mathbb{E}\left[\mathrm{Trace}\{\cdot\}\right] = \mathrm{Trace}\{\mathbb{E}[\cdot]\}$ property in $(**)$, and

$$
\begin{aligned}
\mathbf{K}_{\mathbf{x}|Y=\mathbf{y}} \equiv \mathbf{K}_{\mathbf{x}|Y=\mathbf{y}}(\mathbf{y}) &\overset{\triangle}{=} \mathbb{E}_{\mathbf{x}|Y=\mathbf{y}}\left[(\hat{\mathbf{x}}-\mathbf{x})(\hat{\mathbf{x}}-\mathbf{x})^{\mathrm{T}}|Y=\mathbf{y}\right] && (31)\\
&= \int_{\mathbf{x}} (\mathbb{E}\left[\mathbf{x}|\mathbf{y}\right]-\mathbf{x})(\mathbb{E}\left[\mathbf{x}|\mathbf{y}\right]-\mathbf{x})^{\mathrm{T}} f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x} && (32)\\
&= \int_{\mathbf{x}} (\mathbf{x}-\mathbb{E}\left[\mathbf{x}|\mathbf{y}\right])(\mathbf{x}-\mathbb{E}\left[\mathbf{x}|\mathbf{y}\right])^{\mathrm{T}} f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\ d\mathbf{x}. && (33)
\end{aligned}
$$

MSE Performance Evaluation

- Then, (unconditional) minimum mean square error (MMSE) is calculated:

$$\begin{aligned}
\text{MMSE} &\triangleq \mathbb{E}_{\mathbf{x},\mathbf{y}}\left[(\hat{\mathbf{x}}(\mathbf{y})-\mathbf{x})^{\mathrm{T}}(\hat{\mathbf{x}}(\mathbf{y})-\mathbf{x})\right] &(34)\\
&= \mathbb{E}_{\mathbf{x},\mathbf{y}}\left[\text{Trace}\left\{(\hat{\mathbf{x}}(\mathbf{y})-\mathbf{x})^{\mathrm{T}}(\hat{\mathbf{x}}(\mathbf{y})-\mathbf{x})\right\}\right] &(35)\\
&= \text{Trace}\left\{\mathbb{E}_{\mathbf{x},\mathbf{y}}\left[(\hat{\mathbf{x}}(\mathbf{y})-\mathbf{x})(\hat{\mathbf{x}}(\mathbf{y})-\mathbf{x})^{\mathrm{T}}\right]\right\} &(36)\\
&= \text{Trace}\left\{\mathbf{K}_{\mathrm{E}}\right\} = \text{Trace}\left\{\mathbf{K}_{\mathrm{X/Y}}\right\}, &(37)
\end{aligned}$$

where $\text{Trace}(\mathbf{A}\,\mathbf{B}) = \text{Trace}(\mathbf{B}\,\mathbf{A})$ and $\mathbb{E}\left[\text{Trace}\{\cdot\}\right] = \text{Trace}\{\mathbb{E}[\cdot]\}$ properties were again used and $\mathbf{K}_{\mathrm{E}}$ follows:

$$\begin{aligned}
\mathbf{K}_{\mathrm{E}} &\triangleq \int_{\mathbf{y}}\int_{\mathbf{x}} (\mathbf{x}-\mathbb{E}[\mathbf{x}|\mathbf{y}])(\mathbf{x}-\mathbb{E}[\mathbf{x}|\mathbf{y}])^{\mathrm{T}} f_{\mathbf{x},\mathbf{y}}(\mathbf{x},\mathbf{y})\; d\mathbf{x}\; d\mathbf{y} &(38)\\
&= \int_{\mathbf{y}}\int_{\mathbf{x}} (\mathbf{x}-\mathbb{E}[\mathbf{x}|\mathbf{y}])(\mathbf{x}-\mathbb{E}[\mathbf{x}|\mathbf{y}])^{\mathrm{T}} f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) f_{\mathbf{y}}(\mathbf{y})\; d\mathbf{x}\; d\mathbf{y}\\
&= \int_{\mathbf{y}}\left[\int_{\mathbf{x}} (\mathbf{x}-\mathbb{E}[\mathbf{x}|\mathbf{y}])(\mathbf{x}-\mathbb{E}[\mathbf{x}|\mathbf{y}])^{\mathrm{T}} f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})\; d\mathbf{x}\right] f_{\mathbf{y}}(\mathbf{y})\; d\mathbf{y}\\
&\overset{Eq.\ (33)}{=} \int_{\mathbf{y}} \mathbf{K}_{\mathbf{x}|Y=\mathbf{y}}\; f_{\mathbf{y}}(\mathbf{y})\; d\mathbf{y}. &(39)
\end{aligned}$$

Optimum Bayesian Estimator: MAE case

- For minimum absolute error (MAE) loss function:

$$C(\hat{\mathbf{x}}(\mathbf{y}),\mathbf{x}) = ||\hat{\mathbf{x}}-\mathbf{x}||_1 = |\hat{x}_1 - x_1| + |\hat{x}_2 - x_2| + \ldots + |\hat{x}_m - x_m|$$

$$J(\hat{\mathbf{x}}|\mathbf{y}) = \int_{\mathbf{x}} ||\hat{\mathbf{x}}-\mathbf{x}||_1 \; f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) \; d\mathbf{x} \tag{40}$$

- Denoting $\mathrm{sign}(z) = +1$ if $z \geq 0$ and $\mathrm{sign}(z) = -1$, otherwise, the following are calculated:

$$\frac{\partial J(\hat{\mathbf{x}}|\mathbf{y})}{\partial \hat{x}_i} = \int_{\mathbf{x}} \mathrm{sign}(\hat{x}_i - x_i) \; f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) \; d\mathbf{x} \tag{41}$$

$$= \int_{x_i} \mathrm{sign}(\hat{x}_i - x_i) \times$$

$$\left[\int_{x_1} \int_{x_2} \ldots \int_{x_{i-1}} \int_{x_{i+1}} \ldots \int_{x_m} f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) dx_1 \; dx_2 \ldots dx_{i-1} dx_{i+1} \ldots dx_m \right] dx_i$$

$$= \int_{x_i} \mathrm{sign}(\hat{x}_i - x_i) \; f_{x_i|\mathbf{y}}(x_i|\mathbf{y}) \; dx_i . \tag{42}$$

Optimum Bayesian Estimator: MAE case

- Continuing from previous slide,

$$\frac{\partial J(\hat{\mathbf{x}}|\mathbf{y})}{\partial \hat{x}_i} = 0 \Leftrightarrow \int_{x_i} \operatorname{sign}(\hat{x}_i - x_i)\ f_{x_i|\mathbf{y}}(x_i|\mathbf{y})\ dx_i = 0 \Leftrightarrow \tag{43}$$

$$\int_{-\infty}^{\hat{x}_i} \operatorname{sign}(\hat{x}_i - x_i)\ f_{x_i|\mathbf{y}}(x_i|\mathbf{y})\ dx_i + \int_{\hat{x}_i}^{+\infty} \operatorname{sign}(\hat{x}_i - x_i)\ f_{x_i|\mathbf{y}}(x_i|\mathbf{y})\ dx_i = 0 \Leftrightarrow \tag{44}$$

$$\int_{-\infty}^{\hat{x}_i} f_{x_i|\mathbf{y}}(x_i|\mathbf{y})\ dx_i - \int_{\hat{x}_i}^{+\infty} f_{x_i|\mathbf{y}}(x_i|\mathbf{y})\ dx_i = 0 \Leftrightarrow \tag{45}$$

$$I_1 \triangleq \int_{-\infty}^{\hat{x}_i} f_{x_i|\mathbf{y}}(x_i|\mathbf{y})\ dx_i = \int_{\hat{x}_i}^{+\infty} f_{x_i|\mathbf{y}}(x_i|\mathbf{y})\ dx_i \triangleq I_2 \tag{46}$$

- Since $I_1 + I_2 = 1$ and from above $I_1 = I_2$, the Bayesian MAE estimate is the median of the posterior density:

$$\int_{-\infty}^{\hat{x}_i^{\mathrm{MAE}}} f_{x_i|\mathbf{y}}(x_i|\mathbf{y})\ dx_i = \int_{\hat{x}_i^{\mathrm{MAE}}}^{+\infty} f_{x_i|\mathbf{y}}(x_i|\mathbf{y})\ dx_i = 1/2. \tag{47}$$

Optimum Bayesian Estimator: MAE case

- Thus, the $i-$th entry of $\hat{\mathbf{x}}_{\mathrm{MAE}}(\mathbf{y})$ is the median of the posterior density $f_{x_i|\mathbf{y}}(x_i|\mathbf{y})$:

$$\int_{-\infty}^{\hat{x}_i^{\mathrm{MAE}}} f_{x_i|\mathbf{y}}(x_i|\mathbf{y})\; dx_i = \int_{\hat{x}_i^{\mathrm{MAE}}}^{+\infty} f_{x_i|\mathbf{y}}(x_i|\mathbf{y})\; dx_i = 1/2. \qquad (48)$$

Optimum Bayesian Estimator: Notch Cost Function

- For notch cost function $L_\epsilon(\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}) = \begin{cases} 0, & \text{if } ||\mathbf{e}||_\infty < \epsilon \\ 1, & \text{otherwise,} \end{cases}$, where $||\mathbf{e}||_\infty = \max_i |e_i|$, assuming $\mathbf{e} = [e_1 \quad e_2 \ldots e_m]^{\mathrm{T}}$. Thus,

$$J(\hat{\mathbf{x}}|\mathbf{y}) = \int_{\mathbf{x}} L_\epsilon(\hat{\mathbf{x}} - \mathbf{x}) \ f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) \ d\mathbf{x} = \int_{||\hat{\mathbf{x}}-\mathbf{x}||_\infty \geq \epsilon} f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) \ d\mathbf{x}$$
$$= 1 - \int_{||\hat{\mathbf{x}}-\mathbf{x}||_\infty < \epsilon} f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) \ d\mathbf{x} \tag{49}$$

- For $\epsilon \to 0^+$, minimization of $J(\cdot)$ above is equivalent to maximizing the following:

$$\int_{||\hat{\mathbf{x}}-\mathbf{x}||_\infty < \epsilon} f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) \ d\mathbf{x} \approx (2\epsilon)^m \ f_{\mathbf{x}|\mathbf{y}}(\hat{\mathbf{x}}|\mathbf{y}) \tag{50}$$

- In other words, for small ϵ,

$$\hat{\mathbf{x}}(\mathbf{y}) \equiv \hat{\mathbf{x}}_{\mathrm{MAP}}(\mathbf{y}) = \arg\max_{\mathbf{x}} f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}). \tag{51}$$

Optimum Bayesian Estimator: Notch Cost Function

- For notch cost function $L_\epsilon(\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x})$ and small ϵ,

$$\hat{\mathbf{x}}(\mathbf{y}) \equiv \hat{\mathbf{x}}_{\mathrm{MAP}}(\mathbf{y}) = \arg\max_{\mathbf{x}} f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) \tag{52}$$

- ...the Bayesian estimate becomes the mode (i.e., maximum) of the posterior density (MAP estimate).
- It makes sense if the posterior density $f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})$ has a single dominant peak, or multiple peaks of the same size.
- Example: for jointly Gaussian $\mathbf{x}, \mathbf{y}$, conditional mean, median and mode coincide, i.e.,

$$\hat{\mathbf{x}}_{\mathrm{MSE}}(\mathbf{y}) = \hat{\mathbf{x}}_{\mathrm{MAE}}(\mathbf{y}) = \hat{\mathbf{x}}_{\mathrm{MAP}}(\mathbf{y}) = \mathbb{E}\left[\mathbf{x}|\mathbf{y}\right].$$

References

[1] Bernard C. Levy, Principles of Signal Detection and Parameter Estimation, Springer 2008.
[2] Instructor notes.

Thank you!

Detection & Estimation Theory: Lectures 11-12

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Outline

- Examples of Bayesian Estimation
- Properties of Bayesian MSE Estimator
- MMSE Estimation in Linear Gaussian Systems

Bayesian Estimation Example 1

- The following (exponential) p.d.f.s are given: $f_{y|x}(y|x) = xe^{-xy}u(y), f(x) = ae^{-ax}u(x)$
- Need to compute $f_y(y)$ to compute $f_{x|y}(x|y)$:

$$
\begin{aligned}
f(x,y) = f(y|x)f(x) = axe^{-(a+y)x}u(x)u(y) \Rightarrow \\
f(y) = \int_0^{+\infty} axe^{-(a+y)x}u(y)dx = \frac{a}{(y+a)^2}u(y) \\
\Rightarrow f(x|y) = \frac{f_{x,y}(x,y)}{f(y)} = (y+a)^2 xe^{-(y+a)x}u(x)
\end{aligned}
$$

- MSE estimator:

$$
\hat{x}_{\text{MSE}}(y) = \int_{-\infty}^{+\infty} xf(x|y)dx = \cdots = \frac{2}{y+a}
$$

Bayesian Estimation Example 1

- MAE estimator:

$$\frac{1}{2} = \int_{-\infty}^{\hat{x}} f(x|y)dx = (y+a)^2 \int_0^{\hat{x}} xe^{-(y+a)x}dx = \cdots =$$

$$= \left[1 + (a+y)\hat{x}\right] e^{-(a+y)\hat{x}} \overset{c=(a+y)\hat{x}}{=} (1+c)e^{-c} \Rightarrow$$

$$e^c = 2(1+c) \Rightarrow c = \ln\left[2\left(1+c\right)\right] \Rightarrow c \approx 1.68 \Rightarrow$$

$$\hat{x}_{\text{MAE}}(y) = \frac{c}{a+y} = \frac{1.68}{y+a}$$

- MAP estimator:

$$\frac{\partial f_{x|y}(x|y)}{\partial x} = (y+a)^2 e^{-(y+a)x} + (y+a)^2 x e^{-(y+a)x}(-(y+a)) = 0 \Rightarrow$$

$$(y+a)^2 e^{-(y+a)x}\left(1-(y+a)x\right) = 0 \Rightarrow$$

$$\hat{x}_{\text{MAP}} = \frac{1}{y+a}$$

Bayesian Estimation Example 2

- $\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n$ jointly Gaussian $\Leftrightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N}(\mathbf{m}, \mathbf{K})$

$$\mathbf{m} \triangleq \begin{bmatrix} \mathbb{E}[\mathbf{x}] \\ \mathbb{E}[\mathbf{y}] \end{bmatrix} = \begin{bmatrix} \mathbf{m}_x \\ \mathbf{m}_y \end{bmatrix}, \mathbf{K} \triangleq \mathbb{E}\left\{ \begin{bmatrix} \mathbf{x} - \mathbf{m}_x \\ \mathbf{y} - \mathbf{m}_y \end{bmatrix} \begin{bmatrix} (\mathbf{x} - \mathbf{m}_x)^{\mathrm{T}} & (\mathbf{y} - \mathbf{m}_y)^{\mathrm{T}} \end{bmatrix} \right\}$$
$$= \begin{bmatrix} \mathbf{K}_X & \mathbf{K}_{XY} \\ \mathbf{K}_{YX} & \mathbf{K}_Y \end{bmatrix}, \mathbf{K}_{XY} = \mathbf{K}_{YX}^{\mathrm{T}}$$

- $\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n$ jointly Gaussian $\Rightarrow$

$$f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) = \mathcal{N}\left(\mathbf{m}_{X|Y}, \mathbf{K}_{X|Y}\right)$$

$$\begin{aligned} \mathbf{m}_{X|Y} &= \mathbf{m}_x + \mathbf{K}_{XY}\mathbf{K}_Y^{-1}(\mathbf{y} - \mathbf{m}_y) \\ \mathbf{K}_{X|Y} &= \mathbf{K}_X - \mathbf{K}_{XY}\mathbf{K}_Y^{-1}\mathbf{K}_{YX} \\ \hat{\mathbf{x}}_{\mathrm{MSE}}(\mathbf{y}) &= \mathbb{E}[\mathbf{x}|\mathbf{y}] \equiv \mathbf{m}_{X|Y} = \mathbf{m}_x + \mathbf{K}_{XY}\mathbf{K}_Y^{-1}(\mathbf{y} - \mathbf{m}_y) \\ \mathrm{MMSE} &\triangleq \mathbb{E}\left[||\mathbf{x} - \hat{\mathbf{x}}_{\mathrm{MSE}}||_2^2\right] = \mathrm{Trace}\left(\mathbf{K}_{X|Y}\right) \end{aligned}$$

Bayesian Estimation Example 2

- The three estimates coincide (conditional mean, median, maximum):

$$\hat{\mathbf{x}}_{\mathrm{MSE}}(\mathbf{y}) = \hat{\mathbf{x}}_{\mathrm{MAE}}(\mathbf{y}) = \hat{\mathbf{x}}_{\mathrm{MAP}}(\mathbf{y}) = \mathbf{m}_x + \mathbf{K}_{XY}\mathbf{K}_Y^{-1}(\mathbf{y} - \mathbf{m}_y).$$

Properties of MSE Estimator

1. **Orthogonality Property**:

$$\mathbb{E}\Big[(\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{y}])\ g(\mathbf{y}) \Big] = \mathbf{0} \tag{1}$$

- The above states that the expected value of the inner product of the error vector with any function of the measurements is always zero.
- This property is just an expression of the fact that the conditional mean $\mathbb{E}[\mathbf{x}|\mathbf{y}]$ extracts all the information in $\mathbf{y}$ that can be used to reduce the MSE.
- Notice that $g(\mathbf{y})$ can be scalar or (row) vector.

Proof.

$$\textit{1st method: } \mathbb{E}\left[(\mathbf{x}-\mathbb{E}[\mathbf{x}|\mathbf{y}])\ \ g(\mathbf{y})\right] = \mathbb{E}\left[\mathbf{x}\ g(\mathbf{y})\right] - \mathbb{E}\left[\mathbb{E}[\mathbf{x}|\mathbf{y}]g(\mathbf{y})\right]$$
$$= \mathbb{E}\left[\mathbf{x}\ g(\mathbf{y})\right] - \mathop{\mathbb{E}}_{Y}\left[\mathop{\mathbb{E}}_{X|Y}\left[\mathbf{x}\ g(\mathbf{y})|\mathbf{y}\right]\right]$$
$$\overset{(*)}{=} \mathbb{E}\left[\mathbf{x}\ g(\mathbf{y})\right] - \mathbb{E}\left[\mathbf{x}\ g(\mathbf{y})\right] = \mathbf{0}$$

where at step $(*)$ the law of iterated expectation was used:

$$E[h(x,y)] = \mathop{E}_{Y}\mathop{E}_{X|Y}\left[h(x,y)|y\right] \Leftrightarrow$$
$$\int\int h(x,y)f(x,y)\ dx\ dy = \int_y\int_x h(x,y)\ f(x|y)\ dx\ f(y)\ dy$$

$$\textit{2nd method: } \mathop{\mathbb{E}}_{X,Y}[\mathbf{x}\ g(\mathbf{y})] = \mathop{\mathbb{E}}_{Y}\left\{\mathop{\mathbb{E}}_{X|Y}\left[\mathbf{x}\ g(\mathbf{y})|\ \mathbf{y}\right]\right\} = \mathop{\mathbb{E}}_{Y}\left[\mathop{\mathbb{E}}_{X|Y}\left[\mathbf{x}|\mathbf{y}\right]\ \ g(\mathbf{y})\right]$$

$\square$

Properties of MSE Estimator

2. **Uniqueness Property**:

   $\mathbb{E}[\mathbf{x}|\mathbf{y}]$ is the **unique** vector function $\in \mathbb{R}^m$ that adheres to the orthogonality property.

Proof.

Suppose that $\mathbf{h}(\mathbf{y})$ is another function with $\mathbb{E}\left[(\mathbf{x}-\mathbf{h}(\mathbf{y}))\,g(\mathbf{y})\right]=\mathbf{0}$ for all functions $g(\cdot)$. Then, it follows:

$$
\begin{aligned}
\mathbb{E}\left[||\mathbb{E}[\mathbf{x}|\mathbf{y}]-\mathbf{h}(\mathbf{y})||_2^2\right] &= \mathbb{E}\left[\big(\underbrace{\mathbb{E}(\mathbf{x}|\mathbf{y})-\mathbf{h}(\mathbf{y})}_{g(\mathbf{y})}\big)^{\mathrm{T}}\big(\mathbb{E}[\mathbf{x}|\mathbf{y}]-\mathbf{x}+\mathbf{x}-\mathbf{h}(\mathbf{y})\big)\right] \\
&= \mathbb{E}\left[g^{\mathrm{T}}(\mathbf{y})\,(\mathbb{E}[\mathbf{x}|\mathbf{y}]-\mathbf{x})+g^{\mathrm{T}}(\mathbf{y})\,(\mathbf{x}-\mathbf{h}(\mathbf{y}))\right] \\
&= \mathbf{0}+\mathbf{0}\ \text{(due to orthogonality principle)} \Rightarrow \\
\mathbb{E}[\mathbf{x}|\mathbf{y}] &= \mathbf{h}(\mathbf{y}),
\end{aligned}
$$

since at the last step, the expected value of a non-negative random variable is zero only when the variable is always zero. □

Properties of MSE Estimator

3. **Variance reduction**:

   if $\mathbf{K}_X = \mathbb{E}[(\mathbf{x} - \mathbf{m}_x)(\mathbf{x} - \mathbf{m}_x)^{\mathrm{T}}]$

   and

   $$\mathbf{K}_E = \mathbb{E}\left[\left(\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{y}]\right)\left(\mathbf{x} - \mathbb{E}(\mathbf{x}|\mathbf{y})\right)^{\mathrm{T}}\right] \equiv \mathbf{K}_{X|Y}$$

   then

   - $\mathbf{K}_E \leq \mathbf{K}_X$ i.e., $\mathbf{K}_X - \mathbf{K}_E$ is positive semi-definite,
   - $\mathbf{K}_E = \mathbf{K}_x$ if and only if (iff) $\mathbf{m}_x = \mathbb{E}[\mathbf{x}|\mathbf{y}]$ i.e., knowledge of the observation $\mathbf{y}$ does not improve the estimate of $\mathbf{x}$.

Properties of MSE Estimator

Proof.

$\mathbf{x} - \mathbf{m}_x = \mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{y}] + \mathbb{E}[\mathbf{x}|\mathbf{y}] - \mathbf{m}_x$, therefore

$$\begin{aligned}
\mathbf{K}_X = \mathbf{K}_E &+ \mathbb{E}\left[\big(\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{y}]\big)\big(\mathbb{E}[\mathbf{x}|\mathbf{y}] - \mathbf{m}_x\big)^{\mathrm{T}}\right] \\
&+ \mathbb{E}\left[\big(\mathbb{E}[\mathbf{x}|\mathbf{y}] - \mathbf{m}_x\big)\big(\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{y}]^{\mathrm{T}}\big)\right] \\
&+ \mathbb{E}\Big[\big(\underbrace{\mathbb{E}[\mathbf{x}|\mathbf{y}] - \mathbf{m}_x}_{\Delta(\mathbf{y})}\big)\big(\mathbb{E}[\mathbf{x}|\mathbf{y} - \mathbf{m}_x]\big)^{\mathrm{T}}\Big] \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
\mathbf{K}_X = \mathbf{K}_E &+ \mathbb{E}\Big[\big(\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{y}]\big)\Delta(\mathbf{y})^{\mathrm{T}}\Big] + \mathbb{E}\Big[\Delta(\mathbf{y})\big(\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{y}]\big)^{\mathrm{T}}\Big] \\
&+ \underbrace{\mathbb{E}\left[\Delta(\mathbf{y})\Delta(\mathbf{y})^{\mathrm{T}}\right]}_{\mathbf{K}_\Delta} = \mathbf{K}_E + \mathbf{K}_\Delta \Leftrightarrow
\end{aligned}$$

$\mathbf{K}_X - \mathbf{K}_E = \mathbf{K}_\Delta = \mathbb{E}[\Delta(\mathbf{y})\Delta(\mathbf{y})^{\mathrm{T}}] \geq 0$ since

$\mathbf{z}^{\mathrm{T}}\mathbb{E}\left[\Delta(\mathbf{y})\Delta(\mathbf{y})^{\mathrm{T}}\right]\mathbf{z} = \mathbb{E}\Big[\underbrace{\mathbf{z}^{\mathrm{T}}\Delta\mathbf{y}}_{\mathbf{z}_0^{\mathrm{T}}}\Delta\mathbf{y}^{\mathrm{T}}\mathbf{z}\Big] = \mathbb{E}\left[||\mathbf{z}_0||_2^2\right] \geq 0$ □

Properties of MSE Estimator

...proof continued.

- $\mathbf{K}_\Delta = \mathbf{0}$ (all elements zero) $\Rightarrow$ Trace$(\mathbf{K}_\Delta) = 0$ and the following holds:

$$\begin{aligned}\mathrm{Trace}(\mathbf{K}_\Delta) &= \mathrm{Trace}\left[\mathbb{E}[\Delta\,\Delta^{\mathrm{T}}]\right] = \mathbb{E}[\mathrm{Trace}(\Delta\,\Delta^{\mathrm{T}})]\\ &= \mathbb{E}[\Delta^{\mathrm{T}}\,\Delta] = \mathbb{E}[||\Delta||_2^2] = 0 \Rightarrow\\ \Delta \equiv \Delta(\mathbf{y}) = \mathbf{0} &\Rightarrow \mathbb{E}[\mathbf{x}|\mathbf{y}] = \mathbf{m}_x\end{aligned}$$

- For the other direction, i.e., $\Delta(\mathbf{y}) = \mathbf{0} \Rightarrow \mathbf{K}_\Delta = \mathbf{0}$ the proof is trivial.

$\square$

MMSE Estimation in Linear Gaussian Systems

- Let $\mathbf{x} \in \mathbb{R}^{D_x}$ to be estimated and $\mathbf{y} \in \mathbb{R}^{D_y}$ with

  $$p(\mathbf{x}) \sim \mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x), p(\mathbf{y}|\mathbf{x}) \sim \mathcal{N}(\mathbf{A}\mathbf{x} + \mathbf{b}, \boldsymbol{\Sigma}_y)$$

  where $\mathbf{A}$ (deterministic) $D_y \times D_x$ real matrix, $\mathbf{b} \in \mathbb{R}^{D_y}$ and $\boldsymbol{\mu}_y = \mathbb{E}[\boldsymbol{y}] = \boldsymbol{A}\boldsymbol{\mu}_x + \boldsymbol{b}$.

  Then, $p(\mathbf{x}|\mathbf{y}) \sim \mathcal{N}(\boldsymbol{\mu}_{x|y}, \boldsymbol{\Sigma}_{x|y})$ with
  $\boldsymbol{\mu}_{x|y} = \boldsymbol{\Sigma}_{x|y}\left[\mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}(\mathbf{y} - \mathbf{b}) + \boldsymbol{\Sigma}_x^{-1}\boldsymbol{\mu}_x\right]$
  and
  $\boldsymbol{\Sigma}_{x|y} = \left(\boldsymbol{\Sigma}_x^{-1} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}\mathbf{A}\right)^{-1}$

- Thus,

  $$\hat{\mathbf{x}}_{\mathrm{MMSE}}(\mathbf{y}) \equiv \mathbb{E}[\mathbf{x}|\mathbf{y}] \equiv \boldsymbol{\mu}_{x|y} = \boldsymbol{\Sigma}_{x|y}\left[\mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}(\mathbf{y} - \mathbf{b}) + \boldsymbol{\Sigma}_x^{-1}\boldsymbol{\mu}_x\right].$$

MMSE Estimation in Linear Gaussian Systems

Proof (1/4):

$$p(\mathbf{x},\mathbf{y}) = p(\mathbf{y}|\mathbf{x})p(\mathbf{x}) \Rightarrow$$

$$\log p(\mathbf{x},\mathbf{y}) = \log p(\mathbf{y}|\mathbf{x}) + \log p(\mathbf{x}) =$$

$$= -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_x)^{\mathrm{T}}\boldsymbol{\Sigma}_x^{-1}(\mathbf{x}-\boldsymbol{\mu}_x) - \frac{1}{2}(\mathbf{y}-\mathbf{A}\mathbf{x}-\mathbf{b})^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}(\mathbf{y}-\mathbf{A}\mathbf{x}-\mathbf{b}) +$$

$$+ \text{constant terms}$$

$$= -\frac{1}{2}\mathbf{x}^{\mathrm{T}}\boldsymbol{\Sigma}_x^{-1}\mathbf{x} - \frac{1}{2}\mathbf{y}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}\mathbf{y} - \frac{1}{2}(\mathbf{A}\mathbf{x})^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}(\mathbf{A}\mathbf{x}) + \mathbf{y}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}(\mathbf{A}\mathbf{x}) +$$

$$+ \text{ linear terms } + \text{ constant terms}$$

$$= -\frac{1}{2}\mathbf{x}^{\mathrm{T}}\boldsymbol{\Sigma}_x^{-1}\mathbf{x} - \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}\mathbf{A}\mathbf{x} - \frac{1}{2}\mathbf{y}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}\mathbf{y} + +\mathbf{y}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}(\mathbf{A}\mathbf{x}) +$$

$$+ \text{ linear terms } + \text{ constant terms}$$

Continue proof (2/4):

$$= -\frac{1}{2}\begin{bmatrix}\mathbf{x}^{\mathrm{T}} & \mathbf{y}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma}_x^{-1} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}\mathbf{A} & -\mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1} \\ -\boldsymbol{\Sigma}_y^{-1}\mathbf{A} & \boldsymbol{\Sigma}_y^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ \mathbf{y}\end{bmatrix} + \text{ constant terms}$$

$$= -\frac{1}{2}\begin{bmatrix}\mathbf{x}^{\mathrm{T}} & \mathbf{y}^{\mathrm{T}}\end{bmatrix}\boldsymbol{\Sigma}^{-1}\begin{bmatrix}\mathbf{x} \\ \mathbf{y}\end{bmatrix} \Rightarrow$$

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix}\boldsymbol{\Sigma}_x^{-1} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}\mathbf{A} & -\mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1} \\ -\boldsymbol{\Sigma}_y^{-1}\mathbf{A} & \boldsymbol{\Sigma}_y^{-1}\end{bmatrix} \triangleq \boldsymbol{\Lambda} = \begin{bmatrix}\boldsymbol{\Lambda}_{xx} & \boldsymbol{\Lambda}_{xy} \\ \boldsymbol{\Lambda}_{yx} & \boldsymbol{\Lambda}_{yy}\end{bmatrix} \tag{2}$$

Useful (for the proof) Theorem

Continue proof (3/4): The following theorem will be utilized; its proof will be given in the problem sets and can be found in various textbooks, e.g., Chapter 4 in *"Machine Learning, a Probabilistic Perspective" by Kevin Murphy*):

- Assume $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$ Gaussian vector, i.e. $\mathbf{x}_1, \mathbf{x}_2$ jointly Gaussians, with $\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \boldsymbol{\Sigma} \triangleq \mathbb{E}\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\right] = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \boldsymbol{\Lambda} \triangleq \boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{\Lambda}_{11} & \boldsymbol{\Lambda}_{12} \\ \boldsymbol{\Lambda}_{21} & \boldsymbol{\Lambda}_{22} \end{bmatrix}$ $(\mathbf{A^{**}})$

Then:

$$p(\mathbf{x}_1|\mathbf{x}_2) = \mathcal{N}(\boldsymbol{\mu}_{1|2}, \boldsymbol{\Sigma}_{1|2}),\ p(\mathbf{x}_1) = \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}),\ p(\mathbf{x}_2) = \mathcal{N}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22}),$$

$$\begin{aligned} \boldsymbol{\mu}_{1|2} &= \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2) \\ &= \boldsymbol{\mu}_1 - \boldsymbol{\Lambda}_{11}^{-1}\boldsymbol{\Lambda}_{12}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2) = \boldsymbol{\Lambda}_{11}^{-1}[\boldsymbol{\Lambda}_{11}\boldsymbol{\mu}_1 - \boldsymbol{\Lambda}_{12}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2)] \\ &= \boldsymbol{\Sigma}_{1|2}[\boldsymbol{\Lambda}_{11}\boldsymbol{\mu}_1 - \boldsymbol{\Lambda}_{12}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2)] \qquad (3) \\ \boldsymbol{\Sigma}_{1|2} &= \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} = \boldsymbol{\Lambda}_{11}^{-1} \qquad (4) \end{aligned}$$

MMSE Estimation in Linear Gaussian Systems

Continue proof (4/4).

Thus, from ($\mathbf{A}$**) and Eq. (4),

$$\boldsymbol{\Sigma}_{x|y} = \boldsymbol{\Lambda}_{xx}^{-1} = (\boldsymbol{\Sigma}_x^{-1} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}\mathbf{A})^{-1}$$

From from ($\mathbf{A}$**), Eq. (3), and $\boldsymbol{\mu}_y = \mathbf{A}\boldsymbol{\mu}_x + \boldsymbol{b}$,

$$\begin{aligned}
\boldsymbol{\mu}_{x|y} &= \boldsymbol{\Sigma}_{x|y}\big[\boldsymbol{\Sigma}_{x|y}^{-1}\boldsymbol{\mu}_x - \boldsymbol{\Lambda}_{xy}(\boldsymbol{y} - \boldsymbol{\mu}_y)\big] \\
&= \boldsymbol{\Sigma}_{x|y}\big[(\boldsymbol{\Sigma}_x^{-1} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}\mathbf{A})\boldsymbol{\mu}_x + \mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}(\boldsymbol{y} - \mathbf{A}\boldsymbol{\mu}_x - \boldsymbol{b})\big] \\
&= \boldsymbol{\Sigma}_{x|y}\big[\boldsymbol{\Sigma}_x^{-1}\boldsymbol{\mu}_x + \mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}\mathbf{A}\boldsymbol{\mu}_x + \mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}(\boldsymbol{y} - \boldsymbol{b}) - \mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}\mathbf{A}\boldsymbol{\mu}_x\big] \\
&= \boldsymbol{\Sigma}_{x|y}\big[\boldsymbol{\Sigma}_x^{-1}\boldsymbol{\mu}_x + \mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}(\boldsymbol{y} - \boldsymbol{b})\big]
\end{aligned}$$

□

- Important Remark: for the above $p(\boldsymbol{x})$ and $p(\boldsymbol{y})$, the following can be also shown:

$$p(\boldsymbol{y}) = \mathcal{N}(\mathbf{A}\boldsymbol{\mu}_x + \boldsymbol{b}, \boldsymbol{\Sigma}_y + \mathbf{A}\boldsymbol{\Sigma}_x\mathbf{A}^T)$$

MMSE in Linear Gaussian Systems: Example

$f(\boldsymbol{y}_i|\boldsymbol{x}) = \mathcal{N}(\boldsymbol{x}, \boldsymbol{\Sigma}_y), f(\boldsymbol{x}) = \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$

- We observe $\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_N$, which are i.i.d.
- We would like to estimate $\boldsymbol{x}$ based on $\boldsymbol{y}_0$, where

$$\boldsymbol{y}_0 = \frac{1}{N} \Sigma_{i=1}^{N} \boldsymbol{y}_i \tag{5}$$

  - Notice that $\boldsymbol{y}_0 \sim \mathcal{N}(\boldsymbol{x}, \frac{1}{N}\boldsymbol{\Sigma}_y)$. This can be easily shown by the fact that affine transformation of a Gaussian vector is again a Gaussian vector[1] and the fact that:

$$\boldsymbol{y}_0 = \frac{1}{N} \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}}_{\boldsymbol{B}} \underbrace{\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_N \end{bmatrix}}_{\boldsymbol{y}}$$

[1] if $\mathbf{x} \sim \mathcal{N}(\mathbf{m}, \boldsymbol{\Sigma})$ then $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} \sim \mathcal{N}(\mathbf{A}\mathbf{m} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathrm{T}})$

MMSE in Linear Gaussian Systems: Example

- Thus, $f(\boldsymbol{y}_0|\boldsymbol{x}) = \mathcal{N}(\boldsymbol{x}, \frac{1}{N}\boldsymbol{\Sigma}_y)$, i.e., $\mathbf{A}\boldsymbol{x} + \boldsymbol{b} = \boldsymbol{x} \Rightarrow \mathbf{A} = \mathbf{I}, \boldsymbol{b} = \mathbf{0}$ and $f(\boldsymbol{x}) = \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$.
- With direct application of the above theorem, $f(\boldsymbol{x}|\boldsymbol{y}_0) = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{\mu} = \boldsymbol{\Sigma}\left[\mathbf{I}^{\mathrm{T}}\left(\frac{1}{N}\boldsymbol{\Sigma}_y\right)^{-1}(\boldsymbol{y}_0 - \mathbf{0}) + \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0\right]$$

and

$$\boldsymbol{\Sigma} = \left(\boldsymbol{\Sigma}_0^{-1} + \left(\frac{1}{N}\boldsymbol{\Sigma}_y\right)^{-1}\right)^{-1} = \left(\boldsymbol{\Sigma}_0^{-1} + N\boldsymbol{\Sigma}_y^{-1}\right)^{-1}$$

Therefore,

$$\hat{\boldsymbol{x}}(\boldsymbol{y})_{\mathrm{MSE}} \equiv \boldsymbol{\mu} = \left(\boldsymbol{\Sigma}_0^{-1} + N\boldsymbol{\Sigma}_y^{-1}\right)^{-1}\left(N\boldsymbol{\Sigma}_y^{-1}\boldsymbol{y}_0 + \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0\right).$$

Thank you!

Detection & Estimation Theory: Lecture 13

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Outline

- Bayesian Linear Estimators
  - Derivation
- Remarks
  - Proof on Remark 4
- Linear MSE Estimator Example

Bayesian Linear Estimators: Problem Formulation

- Assume $\mathbf{m}_x = \mathbb{E}[\mathbf{x}]$, $\mathbf{m}_y = \mathbb{E}[\mathbf{y}]$ and the joint covariance matrix $\mathbf{K} = \mathbb{E}\left[\begin{bmatrix}\mathbf{x}-\mathbf{m}_x \\ \mathbf{y}-\mathbf{m}_y\end{bmatrix}\begin{bmatrix}(\mathbf{x}-\mathbf{m}_x)^{\mathrm{T}} & (\mathbf{y}-\mathbf{m}_y)^{\mathrm{T}}\end{bmatrix}\right] = \begin{bmatrix}\mathbf{K}_X & \mathbf{K}_{XY} \\ \mathbf{K}_{YX} & \mathbf{K}_Y\end{bmatrix}$ are known
- $\mathbf{K} > 0$ ($\mathbf{K}^{-1}$ exists); otherwise a non-trivial linear combination in vector $\mathbf{y}-\mathbf{m}_y$ exists, so we could replace observation $\mathbf{y}$ by a vector of smaller dimension!
- $f(\mathbf{y}|\mathbf{x})$ and $f(\mathbf{x})$ are UNKNOWN!
- $\mathbf{y} \in \mathbb{R}^n, \mathbf{x} \in \mathbb{R}^m$
- we are looking for $\hat{\mathbf{x}}_L(\mathbf{y}) = \mathbf{A}\mathbf{y} + \mathbf{b}$ that minimizes $\mathbb{E}\left[\|\mathbf{e}\|_2^2\right]$ (MSE) with the following:
  1. $\mathbf{A}$ a $m \times n$ matrix
  2. $\mathbf{b} \in \mathbb{R}^m$
  3. $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}_L(\mathbf{y}) = \mathbf{x} - A\mathbf{y} - \mathbf{b}$
  4. $\mathbb{E}[\mathbf{e}] \equiv \mathbf{m}_e = \mathbf{m}_x - \mathbf{A}\mathbf{m}_y - \mathbf{b}$

Bayesian Linear Estimators

- Notice that

$$\begin{aligned}\hat{\mathbf{x}}(\mathbf{y}) = \mathbb{E}[\mathbf{x}|\mathbf{y}] &\Rightarrow \mathbb{E}[(\mathbf{x}-\mathbb{E}[\mathbf{x}|\mathbf{y}])\cdot \mathbf{g}(\mathbf{y})] = 0 \text{ (orthogonality property)}\\ &\Rightarrow \mathbb{E}[(\mathbf{x}-\mathbb{E}[\mathbf{x}|\mathbf{y}])] = 0 \qquad (g(\mathbf{y}) = 1)\\ &\Rightarrow \mathbb{E}[\mathbf{e}] = 0\end{aligned}$$

- For $\hat{\mathbf{x}}_L(\mathbf{y}) = \mathbf{A}\mathbf{y} + \mathbf{b} \Rightarrow \mathbb{E}[\mathbf{e}] = \mathbf{m}_x - \mathbf{A}\mathbf{m}_y - \mathbf{b} \neq 0$
- Thus,

$$\mathbf{K}_E \triangleq \mathbb{E}\left[(\mathbf{e}-\mathbf{m}_e)(\mathbf{e}-\mathbf{m}_e)^{\mathrm{T}}\right] \tag{1}$$

$$\mathbf{e}-\mathbf{m}_e = (\mathbf{x}-\mathbf{A}\mathbf{y}-\mathbf{b}) - (\mathbf{m}_x - \mathbf{A}\mathbf{m}_y - \mathbf{b}) \tag{2}$$

$$= \begin{bmatrix}\mathbf{I}_m & -\mathbf{A}\end{bmatrix}\begin{bmatrix}\mathbf{x}-\mathbf{m}_x\\ \mathbf{y}-\mathbf{m}_y\end{bmatrix} \tag{3}$$

- From Eq. (1) and Eq. (3):

$$\mathbf{K}_E = \begin{bmatrix}\mathbf{I}_m & -\mathbf{A}\end{bmatrix}\begin{bmatrix}\mathbf{K}_X & \mathbf{K}_{XY}\\ \mathbf{K}_{YX} & \mathbf{K}_Y\end{bmatrix}\begin{bmatrix}\mathbf{I}_m\\ -\mathbf{A}^{\mathrm{T}}\end{bmatrix}$$

Bayesian Linear Estimators

- MMSE:

$$\mathbb{E}\left[\|\mathbf{e}\|_2^2\right] = \mathbb{E}\left[\mathbf{e}^{\mathrm{T}}\mathbf{e}\right] = \|\mathbf{e}\|_2^2 - \|\mathbf{m}_e\|_2^2 + \|\mathbf{m}_e\|_2^2 \tag{4}$$

$$= \mathbb{E}\left[(\mathbf{e}-\mathbf{m}_e)^{\mathrm{T}}(\mathbf{e}-\mathbf{m}_e)\right] + \|\mathbf{m}_e\|_2^2 \tag{5}$$

$$= \mathrm{Trace}\left(\mathbf{K}_E\right) + \|\mathbf{m}_e\|_2^2 \tag{6}$$

- $\mathbf{K}_E$ depends on $\mathbf{A}$, $\mathbf{m}_e$ depends on $\mathbf{A}$ and $\mathbf{b}$
- Set $\mathbf{m}_e = 0 \Rightarrow \mathbf{m}_x - \mathbf{A}\mathbf{m}_y - \mathbf{b} = 0 \Rightarrow$

$$\mathbf{b} = \mathbf{m}_x - \mathbf{A}\mathbf{m}_y \tag{7}$$

- Now we need to find $\mathbf{A}$. We work as follows:

$$\begin{aligned}
\mathbf{K}_E &= \begin{bmatrix}\mathbf{I}_m & -\mathbf{A}\end{bmatrix}\begin{bmatrix}\mathbf{K}_X & \mathbf{K}_{XY}\\ \mathbf{K}_{YX} & \mathbf{K}_{\mathbf{Y}}\end{bmatrix}\begin{bmatrix}\mathbf{I}_m\\ -\mathbf{A}^{\mathrm{T}}\end{bmatrix}\\
&= \begin{bmatrix}\mathbf{K}_X - \mathbf{A}\mathbf{K}_{YX} & \mathbf{K}_{XY} - \mathbf{A}\mathbf{K}_Y\end{bmatrix}\begin{bmatrix}\mathbf{I}_m\\ -\mathbf{A}^{\mathrm{T}}\end{bmatrix}\\
&= \mathbf{K}_X - \mathbf{A}\mathbf{K}_{YX} - \mathbf{K}_{XY}\mathbf{A}^{\mathrm{T}} + \mathbf{A}\mathbf{K}_Y\mathbf{A}^{\mathrm{T}}\\
&= \mathbf{K}_{XY}\mathbf{K}_Y^{-1}\mathbf{K}_{YX} - \mathbf{K}_{XY}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{K}_{YX} + \mathbf{A}\mathbf{K}_Y\mathbf{A}^{\mathrm{T}} + \mathbf{S}\\
&= \left(\mathbf{K}_{XY} - \mathbf{A}\mathbf{K}_Y\right)\left(\left(\mathbf{K}_{XY}\mathbf{K}_Y^{-1}\right)^{\mathrm{T}} - \mathbf{A}^{\mathrm{T}}\right) + \mathbf{S}\\
&= \left(\mathbf{K}_{XY}\mathbf{K}_Y^{-1} - \mathbf{A}\right)\mathbf{K}_Y\left(\mathbf{K}_{XY}\mathbf{K}_Y^{-1} - \mathbf{A}\right)^{\mathrm{T}} + \mathbf{S}
\end{aligned}$$

Bayesian Linear Estimators

- Schur complement $\mathbf{S} \triangleq \mathbf{K}_X - \mathbf{K}_{XY}\mathbf{K}_Y^{-1}\mathbf{K}_{YX}$
  - Schur complement of $\mathbf{K}_\mathbf{Y}$ in $\mathbf{K}$ plays a role in evaluating the determinant and the inverse of block matrices.
  - Schur complement is constant and known and does not depend on $\mathbf{A}$

  Thus, $\mathrm{Trace}\left(\mathbf{K}_E\right) =$ $\mathrm{Trace}\left(\left(\mathbf{K}_{XY}\mathbf{K}_Y^{-1} - \mathbf{A}\right)\mathbf{K}_Y\left(\mathbf{K}_{XY}\mathbf{K}_Y^{-1} - \mathbf{A}\right)^{\mathrm{T}}\right) + \mathrm{Trace}\left(\mathbf{S}\right)$, since $\mathrm{Trace}\left(\mathbf{A}+\mathbf{B}\right) = \mathrm{Trace}\left(\mathbf{A}\right) + \mathrm{Trace}\left(\mathbf{B}\right)$.
- $\mathrm{Trace}\left(\left(\mathbf{K}_{XY}\mathbf{K}_Y^{-1} - \mathbf{A}\right)\mathbf{K}_Y\left(\mathbf{K}_{XY}\mathbf{K}_Y^{-1} - \mathbf{A}\right)^{\mathrm{T}}\right) \geq 0$, since $\left(\mathbf{K}_{XY}\mathbf{K}_Y^{-1} - \mathbf{A}\right)\mathbf{K}_Y\left(\mathbf{K}_{XY}\mathbf{K}_Y^{-1} - \mathbf{A}\right)^{\mathrm{T}}$ is positive semi-definite. Thus, $\mathrm{Trace}\left(\left(\mathbf{K}_{XY}\mathbf{K}_Y^{-1} - \mathbf{A}\right)\mathbf{K}_Y\left(\mathbf{K}_{XY}\mathbf{K}_Y^{-1} - \mathbf{A}\right)^{\mathrm{T}}\right) = 0 \Leftrightarrow$ $\mathbf{K}_{XY}\mathbf{K}_Y^{-1} = \mathbf{A}$.
- Thus, $\mathrm{Tr}\left(\mathbf{K}_\mathbf{E}\right)$ is minimized iff

$$\mathbf{K}_{XY}\mathbf{K}_Y^{-1} = \mathbf{A} \tag{8}$$

- From Eq. (7) and Eq. (8), $\hat{\mathbf{x}}_L\left(\mathbf{y}\right) = \mathbf{m}_x + \mathbf{K}_{XY}\mathbf{K}_Y^{-1}\left(\mathbf{y} - \mathbf{m}_y\right)$.

Remarks

1. $\mathbb{E}\left[\mathbf{e}\right] = \mathbf{m}_e = \mathbf{0}$
2. $\hat{\mathbf{x}}_L\left(\mathbf{y}\right) = \mathbf{m}_x + \mathbf{K}_{XY}\mathbf{K}_Y^{-1}\left(\mathbf{y} - \mathbf{m}_y\right) = \hat{\mathbf{x}}_{\mathrm{MSE}}\left(\mathbf{y}\right)$
   for $\mathbf{x}, \mathbf{y}$ jointly Gaussians

   $$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathbf{m}_x \\ \mathbf{m}_y \end{bmatrix}, \begin{bmatrix} \mathbf{K}_X & \mathbf{K}_{XY} \\ \mathbf{K}_{YX} & \mathbf{K}_Y \end{bmatrix} \right)$$

   Linear estimate and MSE estimate coincide for the Gaussian case, i.e., all MSE estimates will necessarily be linear.
3. As promised, linear-least-square estimate $\hat{\mathbf{x}}_L\left(\mathbf{y}\right)$ requires knowledge of first and second moments and not knowledge of $f_{\mathbf{y}|\mathbf{x}}\left(\mathbf{y}|\mathbf{x}\right), f_{\mathbf{x}}\left(\mathbf{x}\right)$.
4. Orthogonality property holds only for linear functions of $\mathbf{y}$:

   $$\mathbb{E}\left[\left(\mathbf{x} - \hat{\mathbf{x}}_L\left(\mathbf{y}\right)\right) \cdot \mathbf{g}\left(\mathbf{y}\right)\right] = \mathbf{0}$$

   for all linear functions $\mathbf{g}\left(\mathbf{y}\right) = g_0 + \mathbf{y}^{\mathrm{T}}\mathbf{g}_1$ where g_0 a real scalar and $\mathbf{g}_1$ a vector in $\mathbb{R}^n$.

Proof.

- Estimation error: $\mathbf{x} - \hat{\mathbf{x}}_L(\mathbf{y}) = \begin{bmatrix} \mathbf{I}_m & -\mathbf{K}_{XY}\mathbf{K}_Y^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x} - \mathbf{m}_x \\ \mathbf{y} - \mathbf{m}_y \end{bmatrix}$
- $\mathbf{g}(\mathbf{y}) = g_0 + \mathbf{m}_y^{\mathrm{T}}\mathbf{g}_1 + (\mathbf{y} - \mathbf{m}_y)^{\mathrm{T}}\mathbf{g}_1$ where $g_0 + \mathbf{m}_y^{\mathrm{T}}\mathbf{g}_1$ is a constant term.
- Therefore,

$$
\begin{aligned}
\mathbb{E}\left[(\mathbf{x} - \hat{\mathbf{x}}_L(\mathbf{y}))\,\mathbf{g}(\mathbf{y})\right] &= \mathbb{E}\left[(\mathbf{x} - \hat{\mathbf{x}}_L(\mathbf{y}))\,(\mathbf{y} - \mathbf{m}_y)^{\mathrm{T}}\,\mathbf{g}_1\right] \\
&= \begin{bmatrix} \mathbf{I}_m & -\mathbf{K}_{XY}\mathbf{K}_Y^{-1} \end{bmatrix} \mathbb{E}\left[\begin{bmatrix} \mathbf{x} - \mathbf{m}_x \\ \mathbf{y} - \mathbf{m}_y \end{bmatrix} \begin{bmatrix} \mathbf{y} & -\mathbf{m}_y \end{bmatrix}^{\mathrm{T}}\right] \mathbf{g}_1 \\
&= \underbrace{\begin{bmatrix} \mathbf{I}_m & -\mathbf{K}_{XY}\mathbf{K}_Y^{-1} \end{bmatrix}}_{m\times(m+n)} \underbrace{\begin{bmatrix} \mathbf{K}_{XY} \\ \mathbf{K}_Y \end{bmatrix}}_{(m+n)\times n} \underbrace{\mathbf{g}_1}_{n\times 1} \\
&= \underbrace{(\mathbf{K}_{XY} - \mathbf{K}_{XY})}_{m\times n} \underbrace{\mathbf{g}_1}_{n\times 1} \\
&= \mathbf{0}
\end{aligned}
$$

$\square$

Remarks (Cont'd)

Proof.

5. $\hat{\mathbf{x}}_L(\mathbf{y})$ is the unique linear estimator that adheres to the orthogonality property.
   - Suppose that $\mathbf{h}(\mathbf{y})$ is another linear estimator with the property $\mathbb{E}\left[(\mathbf{x} - \mathbf{h}(\mathbf{y})) \cdot \mathbf{g}(\mathbf{y})\right] = \mathbf{0}$.
   - Thus
   $\mathbb{E}\left[\|\hat{\mathbf{x}}_L(\mathbf{y}) - \mathbf{h}(\mathbf{y})\|_2^2\right] = \mathbb{E}\left[\mathbf{g}^{\mathrm{T}}(\mathbf{y})(\hat{\mathbf{x}}_L(\mathbf{y}) - \mathbf{x} + \mathbf{x} - \mathbf{h}(\mathbf{y}))\right]$
   $= \mathbb{E}\left[\mathbf{g}^{\mathrm{T}}(\mathbf{y})(\hat{\mathbf{x}}_L(\mathbf{y}) - \mathbf{x})\right] + \mathbb{E}\left[\mathbf{g}^{\mathrm{T}}(\mathbf{y})(\mathbf{x} - \mathbf{h}(\mathbf{y}))\right] = \mathbf{0} + \mathbf{0}$
   $= \mathbf{0}$,
   since $\mathbf{g}(\mathbf{y}) = \hat{\mathbf{x}}_L(\mathbf{y}) - \mathbf{h}(\mathbf{y})$ is linear (because $\hat{\mathbf{x}}_L(\mathbf{y}), \mathbf{h}(\mathbf{y})$ are linear in $\mathbf{y}$).

□

Example

- Assume $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$
  1. $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$
  2. $\mathbf{v}$ uncorrelated with $\mathbf{x}$
  3. $\mathbb{E}\left[\mathbf{v}\right] = 0$ and $\mathbb{E}\left[\mathbf{v}\mathbf{v}^{\mathrm{T}}\right] = \mathbf{R} > 0$, i.e., $\mathbf{R}$ is positive-definite
  4. $\mathbf{H}$ (known) constant matrix
- We need to find the $\hat{\mathbf{x}}_L\left(\mathbf{y}\right)$:
  - $\mathbf{m}_y = \mathbf{H}\mathbf{m}_x$
  - $$\begin{aligned}\mathbf{K}_{YX} &= \mathbb{E}\left[\left(\mathbf{y} - \mathbf{m}_y\right)\left(\mathbf{x} - \mathbf{m}_x\right)^{\mathrm{T}}\right] \\ &= \mathbb{E}\left[\left(\mathbf{H}\left(\mathbf{x} - \mathbf{m}_x\right) + \mathbf{v}\right)\left(\mathbf{x} - \mathbf{m}_x\right)^{\mathrm{T}}\right] \\ &= \mathbf{H}\mathbf{K}_X\end{aligned}$$
    since $\mathbb{E}\left[\mathbf{v}\right] = 0$ and $\mathbf{v}, \mathbf{x}$ are uncorrelated
  - $$\begin{aligned}\mathbf{K}_Y &= \mathbb{E}\left[\left(\mathbf{y} - \mathbf{m}_y\right)\left(\mathbf{y} - \mathbf{m}_y\right)^{\mathrm{T}}\right] \\ &= \mathbb{E}\left[\left(\mathbf{H}\left(\mathbf{x} - \mathbf{m}_x\right) + \mathbf{v}\right)\left(\mathbf{H}\left(\mathbf{x} - \mathbf{m}_x\right) + \mathbf{v}\right)^{\mathrm{T}}\right] \\ &= \mathbf{H}\mathbf{K}_X\mathbf{H}^{\mathrm{T}} + \mathbf{R}\end{aligned}$$

Example

- Therefore,
  - $\hat{\mathbf{x}}_L(\mathbf{y}) = \mathbf{m}_x + \mathbf{K}_X\mathbf{H}^{\mathrm{T}}\left(\mathbf{H}\mathbf{K}_X\mathbf{H}^{\mathrm{T}} + \mathbf{R}\right)^{-1}(\mathbf{y} - \mathbf{H}\mathbf{m}_x)$
  - $\mathbf{K}_L = \mathbf{K}_X - \mathbf{K}_X\mathbf{H}^{\mathrm{T}}\left(\mathbf{H}\mathbf{K}_X\mathbf{H}^{\mathrm{T}} + \mathbf{R}\right)^{-1}\mathbf{H}\mathbf{K}_X \equiv \mathbf{K}_E$
- We can show that

$$\mathbf{K}_L^{-1} = \mathbf{K}_X^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H} \qquad \text{(A)}$$

i.e. $\mathbf{K}_L = \left(\mathbf{K}_X^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right)^{-1}$
("Sherman-Morrison-Woodbury" identity)
Proof:

  - $(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C}^{-1} + \mathbf{D}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{D}\mathbf{A}^{-1}$
  - $(\mathbf{K}_L)^{-1} = \mathbf{K}_X - \mathbf{K}_X\mathbf{H}^{\mathrm{T}}\left(\mathbf{R} + \mathbf{H}\mathbf{K}_X\mathbf{H}^{\mathrm{T}}\right)^{-1}\mathbf{H}\mathbf{K}_X$,
    where $\mathbf{A} = \mathbf{K}_X^{-1}$, $\mathbf{B} = \mathbf{H}^{\mathrm{T}}$, $\mathbf{C} = \mathbf{R}^{-1}$, $\mathbf{D} = \mathbf{H}$

Example

- Denote $\mathbf{G} \overset{\triangle}{=} \mathbf{K}_X \mathbf{H}^{\mathrm{T}} \left(\mathbf{H}\mathbf{K}_X\mathbf{H}^{\mathrm{T}} + \mathbf{R}\right)^{-1} \qquad (\alpha)$
- It is true that $\mathbf{G} = \mathbf{K}_L \mathbf{H}^{\mathrm{T}} \mathbf{R}^{-1} \qquad (\beta)$
- Proof:
  1. $\mathbf{K}_L^{-1} \cdot \mathbf{G} \cdot \left(\mathbf{H}\mathbf{K}_X\mathbf{H}^{\mathrm{T}} + \mathbf{R}\right) \overset{(\beta)}{=} \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1} \left(\mathbf{H}\mathbf{K}_X\mathbf{H}^{\mathrm{T}} + \mathbf{R}\right) \qquad (6)$
  2. $\mathbf{K}_L^{-1} \cdot \mathbf{G} \cdot \left(\mathbf{H}\mathbf{K}_X\mathbf{H}^{\mathrm{T}} + \mathbf{R}\right) \overset{(\alpha)}{=} \left(\mathbf{K}_X^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right) \mathbf{K}_X \mathbf{H}^{\mathrm{T}} \ (7)$
     which is true
  3. (6) = (7) after simple manipulations...
- Also

$$\begin{aligned}
\mathbf{K}_L^{-1} \hat{\mathbf{x}}_L(\mathbf{y}) &= \left(\mathbf{K}_X^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right)\left(\mathbf{m}_x + \mathbf{G} \cdot (\mathbf{y} - \mathbf{H}\mathbf{m}_x)\right) \\
&= \mathbf{K}_X^{-1}\mathbf{m}_x + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\mathbf{m}_x + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y} - \mathbf{H}\mathbf{m}_x) \\
&= \mathbf{K}_X^{-1}\mathbf{m}_x + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{y} \qquad (\mathrm{B})
\end{aligned}$$

- (A), (B) are used in the derivation of Kalman Filter.

Example

- Since
  1. $\mathbf{G} \equiv \mathbf{K}_L \mathbf{H}^{\mathrm{T}} \mathbf{R}^{-1}$
  2. $\hat{\mathbf{x}}_L(\mathbf{y}) = \mathbf{m}_x + \underbrace{\mathbf{K}_X \mathbf{H}^{\mathrm{T}} \left(\mathbf{H}\mathbf{K}_X \mathbf{H}^{\mathrm{T}} + \mathbf{R}\right)^{-1}}_{\mathbf{G}} (\mathbf{y} - \mathbf{H}\mathbf{m}_x)$
  3. $\mathbf{K}_L \equiv \left(\mathbf{K}_X^{-1} + \mathbf{H}^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{H}\right)^{-1}$
- Then $\hat{\mathbf{x}}_L(\mathbf{y}) = \mathbf{m}_x + \left(\mathbf{K}_X^{-1} + \mathbf{H}^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{H}\right)^{-1} \mathbf{H}^{\mathrm{T}} \mathbf{R}^{-1} (\mathbf{y} - \mathbf{H}\mathbf{m}_x)$
- Notice that the above expression is regularly used in various textbooks.

Thank you!

Detection & Estimation Theory: Lectures 14-16

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Outline

- Estimation of Non-random Parameters
  - ML Estimation Examples
- Performance: Cramer-Rao Bound
- Cramer-Rao Bound Derivation
- Existence of Efficient Estimator

Estimation of Non-random Parameters

- Alternative view: $f_{\mathbf{x}}(\mathbf{x})$ not available, $\mathbf{x} \in \mathbb{R}^m$ is viewed as unknown and non-random!
  1. Likelihood function $f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})$ of measurements vector $\mathbf{y}$ when the parameter vector is $\mathbf{x}$.
  2. One simple solution: maximum likelihood (ML) estimate!

$$\hat{\mathbf{x}}_{\mathrm{ML}}(\mathbf{y}) = \arg\max_{\mathbf{x}\in\mathbb{R}^m} f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \tag{1}$$

$$= \arg\max_{\mathbf{x}\in\mathbb{R}^m} \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \tag{2}$$

...the latter (logarithmic) is convenient for p.d.f. in the exponential family (Poisson, Exponential, Gaussian):

$$f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) = \exp\left(\mathbf{x}^{\mathrm{T}}\mathbf{s}(\mathbf{y}) - \mathbf{t}(\mathbf{x})\right)$$

- Bias: $\mathbf{b} \triangleq \mathbb{E}\left[\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y})\right] = \mathbf{x} - \mathbb{E}\left[\hat{\mathbf{x}}(\mathbf{y})\right]$
- Bias of ML estimate may not be zero.

Estimation of Non-random Parameters: Bias

- Bias: $\mathbf{b} \triangleq \mathbb{E}\left[\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y})\right] = \mathbf{x} - \mathbb{E}\left[\hat{\mathbf{x}}(\mathbf{y})\right]$
- ...is the expected value of the error.
- ...is a weak metric, since it does not ensure that for a single measurement vector $\mathbf{y}$ the estimate will offer the true parameter vector $\mathbf{x}$.

Estimation of Non-random Parameters: Examples

- Assume $\mathbf{y} \sim \mathcal{N}\left(A\mathbf{s}, \sigma^2 \mathbf{I}_N\right)$, where $\mathbf{y} \in \mathbb{R}^N$ and $\mathbf{s}, \sigma^2$ known.
- $\hat{A}_{\mathrm{ML}}(\mathbf{y})$?

$$\ln\left[f_{\mathbf{y}|A}(\mathbf{y}|A)\right] = -\frac{1}{2\sigma^2}||\mathbf{y} - A\mathbf{s}||_2^2 + \ln\left[\frac{1}{\sqrt{(2\pi\sigma^2)^N}}\right] \tag{3}$$

$$\Rightarrow \arg\max_{A\in\mathbb{R}} \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) = \arg\min_{A}||\mathbf{y} - A\mathbf{s}||_2^2 \tag{4}$$

$$= \arg\min_{A}\Big(\mathbf{y} - A\mathbf{s})^{\mathrm{T}}(\mathbf{y} - A\mathbf{s}\Big) \tag{5}$$

$$= \arg\min_{A}\left(||\mathbf{s}||_2^2 A^2 - 2\mathbf{s}^{\mathrm{T}}\mathbf{y}A + ||\mathbf{y}||_2^2\right) \tag{6}$$

$$= \frac{\mathbf{s}^{\mathrm{T}}\mathbf{y}}{||\mathbf{s}||_2^2} \tag{7}$$

$$\Rightarrow \hat{A}_{\mathrm{ML}}(\mathbf{y}) = \frac{\mathbf{s}^{\mathrm{T}}\mathbf{y}}{||\mathbf{s}||_2^2}. \tag{8}$$

Estimation of Non-random Parameters: Examples

- Assume $\mathbf{y} = [y_1 \;\; y_2 \dots y_N]^{\mathrm{T}}$ of N independent and identically distributed (i.i.d.) r.v.'s $\{y_k\}$, with $y_k \sim \mathcal{N}(m, v)$.
- Estimation problems:
  1. Estimate m with v known.
  2. Estimate v with m known.
  3. Estimate m, v.
- Check bias of estimate(s).

Estimation of Non-random Parameters: Examples

- Assume $\mathbf{y} = [y_1 \;\; y_2 \ldots y_N]^{\mathrm{T}}$ of N independent and identically distributed (i.i.d.) r.v.'s $\{y_k\}$, with $y_k \sim \mathcal{N}(m, v)$.
- Estimation problem:
  1. Estimate m with v known. Check bias of estimate.

Solution: ...this is the previous example, with $A \equiv m$, $\mathbf{s} = [1 \;\; 1 \ldots 1]^{\mathrm{T}}$ and $\sigma^2 = v$, Thus,

$$\hat{m}_{\mathrm{ML}}(\mathbf{y}) = \frac{\mathbf{s}^{\mathrm{T}}\mathbf{y}}{N} = \frac{\sum_{k=1}^{N} y_k}{N} \tag{9}$$

$$\mathbb{E}\left[\hat{m}_{\mathrm{ML}}(\mathbf{y})\right] = \frac{Nm}{N} = m \text{ (unbiased estimate)} \tag{10}$$

Estimation of Non-random Parameters: Examples

- Estimation problem:
  2. Estimate v with m known. Check bias of estimate.

Solution: $\ln\left[f_{\mathbf{y}|v}(\mathbf{y}|v)\right] = -\frac{N}{2}\ln(2\pi v) - \frac{1}{2v}\sum_{k=1}^{N}(y_k - m)^2$

$$\frac{d}{dv}\ln\left[f_{\mathbf{y}|v}(\mathbf{y}|v)\right] = 0 \Rightarrow \frac{1}{2v}\left[-N + \frac{1}{v}\sum_{k=1}^{N}(y_k - m)^2\right] = 0 \quad (11)$$

$$\Rightarrow \hat{v}_{\mathrm{ML}} = \frac{1}{N}\sum_{k=1}^{N}(y_k - m)^2 \quad (12)$$

- Need to make sure that $\frac{d^2}{dv^2}\ln\left[f_{\mathbf{y}|v}(\mathbf{y}|v)\right] < 0$ at $v = \hat{v}_{\mathrm{ML}}$:

$$\frac{d^2}{dv^2}\ln\left[f_{\mathbf{y}|v}(\mathbf{y}|v)\right] = \ldots = \frac{1}{v^2}\left(\frac{N}{2} - \frac{1}{v}\sum_{k=1}^{N}(y_k - m)^2\right) \quad (13)$$

$$\overset{v=\hat{v}_{\mathrm{ML}}}{=} -\frac{1}{\hat{v}_{\mathrm{ML}}^2}\frac{N}{2} < 0 \quad (14)$$

Estimation of Non-random Parameters: Examples

- Assume $\mathbf{y} = [y_1 \;\; y_2 \dots y_N]^{\mathrm{T}}$ of N independent and identically distributed (i.i.d.) r.v.'s $\{y_k\}$, with $y_k \sim \mathcal{N}(m, v)$.
- Estimation problem:

  2 Estimate v with m known. Check bias of estimate.

$$\hat{v}_{\mathrm{ML}} = \frac{1}{N} \sum_{k=1}^{N} (y_k - m)^2 \tag{15}$$

$$\mathbb{E}\left[\hat{v}_{\mathrm{ML}}\right] = \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}\left[(y_k - m)^2\right] = \frac{Nv}{N} = v \text{ (unbiased estimate).} \tag{16}$$

Estimation of Non-random Parameters: Examples

- Estimation problem:
  3. Estimate m, v. Check bias of estimates.

$$\frac{\partial}{\partial m} \ln \left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v) \right] = 0 \Rightarrow \frac{1}{v} \sum_{k=1}^{N} (y_k - m) = 0 \tag{17}$$

$$\Rightarrow \hat{m}_{\mathrm{ML}} = \frac{1}{N} \sum_{k=1}^{N} y_k \tag{18}$$

$$\frac{\partial}{\partial v} \ln \left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v) \right] = 0 \Rightarrow \frac{1}{2v} \left[-N + \frac{1}{v} \sum_{k=1}^{N} (y_k - m)^2 \right] = 0 \tag{19}$$

$$\Rightarrow \hat{v}_{\mathrm{ML}} = \frac{1}{N} \sum_{k=1}^{N} (y_k - \hat{m}_{\mathrm{ML}})^2 \tag{20}$$

- How do we know that the above maximise the log-likelihood?

Estimation of Non-random Parameters: Examples

- Estimation problem:
  3. Estimate m, v. Check bias of estimates.

$$\hat{m}_{\mathrm{ML}} = \frac{1}{N}\sum_{k=1}^{N} y_k \tag{21}$$

$$\hat{v}_{\mathrm{ML}} = \frac{1}{N}\sum_{k=1}^{N} (y_k - \hat{m}_{\mathrm{ML}})^2 \tag{22}$$

$$g(m,v) \triangleq \ln\left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v)\right] \tag{23}$$

$$\mathbf{H}_g(m,v) = \begin{bmatrix} \frac{\partial^2 g(m,v)}{\partial m^2} & \frac{\partial^2 g(m,v)}{\partial m\,\partial v} \\ \frac{\partial^2 g(m,v)}{\partial v\,\partial m} & \frac{\partial^2 g(m,v)}{\partial v^2} \end{bmatrix} \tag{24}$$

- Need to check that the Hessian matrix $\mathbf{H}_g$ on $g(m,v)$ for $m = \hat{m}_{\mathrm{ML}}$ and $v = \hat{v}_{\mathrm{ML}}$ is non-negative definite (left as an exercise for the reader).

Estimation of Non-random Parameters: Examples

- Estimation problem:

    3 Estimate m, v. Check bias of estimates.

$$\mathbb{E}\left[\hat{v}_{\mathrm{ML}}\right] = \mathbb{E}\left[\frac{1}{N}\sum_{k=1}^{N}(y_k - \hat{m}_{\mathrm{ML}})^2\right] \tag{25}$$

$$= \mathbb{E}\left[\frac{1}{N}\sum_{k=1}^{N} y_k^2\right] + \mathbb{E}\left[\frac{1}{N}\sum_{k=1}^{N}\hat{m}_{\mathrm{ML}}^2\right] - \mathbb{E}\left[\frac{2}{N}\sum_{k=1}^{N}(\hat{m}_{\mathrm{ML}}\, y_k)\right] \tag{26}$$

$$= \ldots = \frac{N-1}{N} v \neq v \text{ (biased estimate)} \tag{27}$$

- That is why numerical packages utilise the following, non-ML, unbiased variance estimate:

$$\frac{1}{N-1}\sum_{k=1}^{N}(y_k - \hat{m}_{\mathrm{ML}})^2$$

Performance of Non-random Parameter Estimation

- Apart from bias, we need the mean square error (MSE) and the corresponding error matrix:

$$\mathbf{C}_{\mathrm{E}} = \mathbb{E}\left[\left(\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y})\right)\left(\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y})\right)^{\mathrm{T}}\right] \tag{28}$$

$$\mathrm{MSE} = \mathrm{Trace}\left(\mathbf{C}_{\mathrm{E}}\right) = \mathbb{E}\left[||\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y})||_2^2\right] \tag{29}$$

- Denote the gradient (vector) $\nabla_x = [\frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \dots \frac{\partial}{\partial x_m}]^{\mathrm{T}}$ and the Hessian (matrix) $\nabla_x \nabla_x^{\mathrm{T}}$.

Non-random Parameter Estimation:Cramer-Rao Bound

- $m \times m$ Fisher Information matrix $\mathbf{J}(\mathbf{x})$ characterises the information in $\mathbf{y} \in \mathbb{R}^n$ about the parameter vector $\mathbf{x} \in \mathbb{R}^n$:

$$\mathbf{J}(\mathbf{x}) \triangleq \mathbb{E}_{\mathbf{y}} \left[\left[\nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \right] \left[\nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \right]^{\mathrm{T}} \right] \tag{30}$$

- It turns out that

$$\mathbf{J}(\mathbf{x}) = -\mathbb{E}_{\mathbf{y}} \left[\underbrace{\nabla_x \nabla_x^{\mathrm{T}} \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})}_{\text{Hessian of the log-likelihood}} \right] \tag{31}$$

Theorem

For any ***unbiased*** *estimator* $\hat{\mathbf{x}}(\mathbf{y})$*, the MSE is bounded from the Cramer-Rao bound, which stems from the diagonal elements of the inverse Fisher Information matrix:*

$$\mathbb{E}\left[||\mathbf{x}_i - \hat{\mathbf{x}}(\mathbf{y})_i||_2^2 \right] \geq \left[\mathbf{J}^{-1}(\mathbf{x}) \right]_{ii}, \tag{32}$$

where $\mathbf{a}_i, \mathbf{A}_{ii}$ *denotes the* $i-th$ *element and* $i-th$ *diagonal element of vector* $\mathbf{a}$ *and matrix* $\mathbf{A}$*, respectively.*

- Assume $\mathbf{y} = [y_1 \;\; y_2 \ldots y_N]^{\mathrm{T}}$, with $\{y_k\}$ i.i.d. and $y_k \sim \mathcal{N}(m, v)$.

$$\ln\left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v)\right] = -\frac{N}{2}\ln(2\pi v) - \frac{1}{2v}\sum_{k=1}^{N}(y_k - m)^2$$

$$\frac{\partial}{\partial m}\ln\left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v)\right] = +\frac{1}{v}\sum_{k=1}^{N}(y_k - m) \tag{33}$$

$$\frac{\partial^2}{\partial m^2}\ln\left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v)\right] = -\frac{N}{v} \tag{34}$$

$$\frac{\partial}{\partial v}\ln\left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v)\right] = -\frac{N}{2v} + \frac{1}{2v^2}\sum_{k=1}^{N}(y_k - m)^2 \tag{35}$$

$$\frac{\partial^2}{\partial v^2}\ln\left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v)\right] = +\frac{N}{2v^2} - \frac{1}{v^3}\sum_{k=1}^{N}(y_k - m)^2 \tag{36}$$

$$\frac{\partial^2}{\partial m\,\partial v}\ln\left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v)\right] = -\frac{1}{v^2}\sum_{k=1}^{N}(y_k - m) \tag{37}$$

Cramer-Rao Bound Example

- ...will use the Hessian version of $\mathbf{J}$. Thus,

$$-\mathbb{E}_{\mathbf{y}}\left[\frac{\partial^2}{\partial m^2}\ln\left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v)\right]\right] = -\mathbb{E}_{\mathbf{y}}\left[-\frac{N}{v}\right] = \frac{N}{v} \tag{38}$$

$$\begin{aligned}-\mathbb{E}_{\mathbf{y}}\left[\frac{\partial^2}{\partial v^2}\ln\left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v)\right]\right] &= -\mathbb{E}_{\mathbf{y}}\left[+\frac{N}{2v^2} - \frac{1}{v^3}\sum_{k=1}^{N}(y_k - m)^2\right] = \\ &= +\frac{N}{2v^2}\end{aligned} \tag{39}$$

$$-\mathbb{E}_{\mathbf{y}}\left[\frac{\partial^2}{\partial m\,\partial v}\ln\left[f_{\mathbf{y}|m,v}(\mathbf{y}|m,v)\right]\right] = -\mathbb{E}_{\mathbf{y}}\left[-\frac{1}{v^2}\sum_{k=1}^{N}(y_k - m)\right] = 0 \tag{40}$$

Cramer-Rao Bound Example

- Now, Fisher Information matrix and its inverse can be calculated:

$$\mathbf{J}(m,v) = \begin{bmatrix} \frac{N}{v} & 0 \\ 0 & \frac{N}{2v^2} \end{bmatrix} \Leftrightarrow \mathbf{J}^{-1}(m,v) = \begin{bmatrix} \frac{v}{N} & 0 \\ 0 & \frac{2v^2}{N} \end{bmatrix} \tag{41}$$

- Therefore, for any unbiased estimate of m, v,

$$\mathbb{E}\left[(m - \hat{m}(\mathbf{y})^2)\right] \geq \frac{v}{N} \tag{42}$$

$$\mathbb{E}\left[(v - \hat{v}(\mathbf{y})^2)\right] \geq \frac{2v^2}{N} \tag{43}$$

Schur Complement Properties

- Before offering the derivation, we first list some basic properties. For any symmetric matrix $\mathbf{M}$ of the following form:[1]

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{C} \end{bmatrix} \tag{44}$$

- If $\mathbf{C}$ is invertible then:
  1. $\mathbf{M} > \mathbf{0}$ iff $\mathbf{C} > \mathbf{0}$ and $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^{\mathrm{T}} > \mathbf{0}$
  2. For $\mathbf{C} > \mathbf{0}$: $\mathbf{M} \geq \mathbf{0} \Leftrightarrow \mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^{\mathrm{T}} \geq \mathbf{0}$
  3. Schur complement $\mathbf{M}|\mathbf{C} \overset{\triangle}{=} \mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^{\mathrm{T}} \Rightarrow \det(\mathbf{M}) = \det(\mathbf{M}|\mathbf{C})\det(\mathbf{C})$
- If $\mathbf{A}$ is invertible then:
  1. $\mathbf{M} > \mathbf{0}$ iff $\mathbf{A} > \mathbf{0}$ and $\mathbf{C} - \mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{B} > \mathbf{0}$
  2. For $\mathbf{A} > \mathbf{0}$: $\mathbf{M} \geq \mathbf{0} \Leftrightarrow \mathbf{C} - \mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{B} \geq \mathbf{0}$
  3. Schur complement $\mathbf{M}|\mathbf{A} \overset{\triangle}{=} \mathbf{C} - \mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{B} \Rightarrow \det(\mathbf{M}) = \det(\mathbf{M}|\mathbf{A})\det(\mathbf{A})$

[1] matrix inequality in the positive semi-definite sense: $\mathbf{A} \geq \mathbf{0} \Leftrightarrow \mathbf{z}^{\mathrm{T}}\mathbf{A}\mathbf{z} \geq 0, \forall \mathbf{z}$

Cramer-Rao Bound Derivation

- Proof: First, we show the two equivalent forms of the Fisher $m \times m$ matrix $\mathbf{J}(\mathbf{x})$:

$$\int f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})d\mathbf{y} = 1 \tag{45}$$

$$\nabla_x \ln\left[f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right] = \frac{1}{f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})}\nabla_x f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \Rightarrow \tag{46}$$

$$\nabla_x^{\mathrm{T}} f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) = \nabla_x^{\mathrm{T}} \ln\left[f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right] f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \tag{47}$$

$$\overset{(45)}{\Rightarrow} \int \nabla_x^{\mathrm{T}} f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})d\mathbf{y} \overset{(47)}{=} \int \nabla_x^{\mathrm{T}} \ln\left[f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right] f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})d\mathbf{y} = \mathbf{0}$$

$$\overset{\nabla_x}{\Rightarrow} \int \nabla_x \nabla_x^{\mathrm{T}} \ln\left[f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right] f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})d\mathbf{y} +$$

$$+ \int \nabla_x \ln\left[f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right] \nabla_x^{\mathrm{T}} \ln\left[f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right] f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})d\mathbf{y} = \mathbf{O} \tag{48}$$

$$\Rightarrow \mathbf{J}_{ij}(\mathbf{x}) = \mathbb{E}_{\mathbf{y}}\left[\frac{\partial}{\partial x_i}\ln\left[f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right] \frac{\partial}{\partial x_j}\ln\left[f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right]\right]$$

$$= -\mathbb{E}_{\mathbf{y}}\left[\frac{\partial^2}{\partial x_i \partial x_j}\ln\left[f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right]\right] \tag{49}$$

Cramer-Rao Bound Derivation

- Next, we define the $2m \times 1$ vector $\mathbf{z}$, corresponding positive semi-definite matrix $\mathbf{C}_z$ and bias $\mathbf{b}(\mathbf{x})$:

$$\mathbf{z} = \begin{bmatrix} \underbrace{\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y})}_{\mathbf{e}} - \mathbf{b}(\mathbf{x}) \\ \\ \nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \end{bmatrix}, \mathbf{C}_z = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} = \mathbb{E}\left[\mathbf{z}\mathbf{z}^{\mathrm{T}}\right] \geq 0 \tag{50}$$

- The following hold:
  1. $\mathbf{C}_{11} = \mathbb{E}\left[(\mathbf{e} - \mathbf{x})(\mathbf{e} - \mathbf{x})^{\mathrm{T}}\right] = \mathbf{C}_{\mathrm{E}} - \mathbf{b}(\mathbf{x})\mathbf{b}^{\mathrm{T}}(\mathbf{x})$.
  2. $\mathbf{C}_{22} = \mathbf{J}(\mathbf{x})$.
  3. $\mathbf{C}_{12} = \mathbf{C}_{21}^{\mathrm{T}} = \mathbb{E}\left[(\mathbf{x} - \hat{\mathbf{x}} - \mathbf{b}(\mathbf{x}))\nabla_x^{\mathrm{T}} \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right] = -\mathbf{I}_m + \nabla_x^{\mathrm{T}}\mathbf{b}(\mathbf{x})$.

- The proof for 3. above follows.

Cramer-Rao Bound Derivation

- Next, we show $\mathbf{C}_{12} = \mathbf{C}_{21}^{\mathrm{T}} = -\mathbf{I}_m + \nabla_x^{\mathrm{T}}\mathbf{b}(\mathbf{x})$.

$$\mathbf{C}_{12} = \mathbf{C}_{21}^{\mathrm{T}} = \mathbb{E}\left[(\mathbf{x} - \hat{\mathbf{x}} - \mathbf{b}(\mathbf{x}))\nabla_x^{\mathrm{T}} \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right] \tag{51}$$

$$= \int_{\mathbf{y}} (\mathbf{x} - \hat{\mathbf{x}} - \mathbf{b}(\mathbf{x}))\underbrace{\nabla_x^{\mathrm{T}} \ln\left[f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right] f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})}_{\nabla_x^{\mathrm{T}} f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})} d\mathbf{y} \tag{52}$$

$$\mathbb{E}\left[(\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y}) - \mathbf{b}(\mathbf{x})\right] = \mathbf{0} \tag{53}$$

$$\Rightarrow \int_{\mathbf{y}} (\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y}) - \mathbf{b}(\mathbf{x}) f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) d\mathbf{y} = 0 \tag{54}$$

$$\overset{\nabla_x^{\mathrm{T}}}{\Rightarrow} \underbrace{\int_{\mathbf{y}} (\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y}) - \mathbf{b}(\mathbf{x}) \nabla_x^{\mathrm{T}} f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) d\mathbf{y}}_{\mathbf{C}_{12}} +$$

$$+ \int_{\mathbf{y}} (\mathbf{I}_m - \nabla_x^{\mathrm{T}}\mathbf{b}(\mathbf{x})) f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) d\mathbf{y} = \mathbf{O} \tag{55}$$

$$\Rightarrow \mathbf{C}_{12} = -\mathbf{I}_m + \nabla_x^{\mathrm{T}}\mathbf{b}(\mathbf{x}) \tag{56}$$

Cramer-Rao Bound Derivation

- In summary:
  1. $\mathbf{C}_{11} = \mathbb{E}\left[(\mathbf{e}-\mathbf{x})(\mathbf{e}-\mathbf{x})^{\mathrm{T}}\right] = \mathbf{C}_{\mathrm{E}} - \mathbf{b}(\mathbf{x})\mathbf{b}^{\mathrm{T}}(\mathbf{x})$.
  2. $\mathbf{C}_{22} = \mathbf{J}(\mathbf{x})$.
  3. $\mathbf{C}_{12} = -\mathbf{I}_m + \nabla_x^{\mathrm{T}}\mathbf{b}(\mathbf{x})$.

$$\mathbf{z} = \begin{bmatrix} \underbrace{\mathbf{x}-\hat{\mathbf{x}}(\mathbf{y})}_{\mathbf{e}} - \mathbf{b}(\mathbf{x}) \\ \\ \nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \end{bmatrix}, \mathbf{C}_z = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} = \mathbb{E}\left[\mathbf{z}\mathbf{z}^{\mathrm{T}}\right] \geq 0 \tag{57}$$

- We assume that positive semi-definite $\mathbf{J}(\mathbf{x})$ is invertible, i.e., it is positive definite. We also known that $\mathbf{C}_z$ is positive semi-definite.
- Thus, the Schur complement is also positive semi-definite:

$$\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}^{\mathrm{T}} \geq \mathbf{0} \tag{58}$$

Cramer-Rao Bound Derivation

- In summary:
  1. $\mathbf{C}_{11} = \mathbb{E}\left[(\mathbf{e}-\mathbf{x})(\mathbf{e}-\mathbf{x})^{\mathrm{T}}\right] = \mathbf{C}_{\mathrm{E}} - \mathbf{b}(\mathbf{x})\mathbf{b}^{\mathrm{T}}(\mathbf{x})$.
  2. $\mathbf{C}_{22} = \mathbf{J}(\mathbf{x})$.
  3. $\mathbf{C}_{12} = -\mathbf{I}_m + \nabla_x^{\mathrm{T}}\mathbf{b}(\mathbf{x})$.

$$\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}^{\mathrm{T}} \geq \mathbf{0} \Rightarrow \tag{59}$$

$$\mathbf{C}_{\mathrm{E}} - \mathbf{b}(\mathbf{x})\mathbf{b}^{\mathrm{T}}(\mathbf{x}) - \left(\mathbf{I}_m - \nabla_x^{\mathrm{T}}\mathbf{b}(\mathbf{x})\right)\mathbf{J}^{-1}(\mathbf{x})\left(\mathbf{I}_m - \nabla_x^{\mathrm{T}}\mathbf{b}(\mathbf{x})\right)^{\mathrm{T}} \geq \mathbf{0} \tag{60}$$

- For unbiased estimator, i.e., $\mathbf{b}(\mathbf{x}) = \mathbf{0}$, the above is simplified to:

$$\mathbf{C}_{\mathrm{E}} - \mathbf{J}^{-1}(\mathbf{x}) \geq \mathbf{0}$$

which completes the proof, if we consider that the diagonal elements of a positive semi-definite matrix are non-negative. ■

Existence of Efficient Estimator - Some Properties

- From Cramer-Rao proof, the following positive semi-definite matrix was utilized:

$$\mathbf{C}_z = \begin{bmatrix} \mathbf{C}_{11} = \mathbf{C}_{\mathrm{E}} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} = \mathbf{J}(\mathbf{x}) \end{bmatrix} = \mathbb{E}\left[\mathbf{z}\mathbf{z}^{\mathrm{T}}\right] \geq 0 \Rightarrow \tag{61}$$

$$\mathbf{C}_z^{-1} = \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \mathbf{z} = \begin{bmatrix} \mathbf{x} - \hat{\mathbf{x}}(\mathbf{y}) - \mathbf{b} \\ \nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \end{bmatrix}, \tag{62}$$

- The following properties hold:
  1. $\mathbf{S} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}$.
  2. $\mathbf{C}_z > 0 \Rightarrow \mathbf{S} > 0$.
  3. $\mathbf{C}_z \geq 0 \Rightarrow \mathbf{S} \geq 0$.
  4. $\mathbf{S} = \mathbf{0} \Rightarrow 2m \times 2m$ matrix $\mathbf{C}_z$ is of rank m.[2]

[2] $\mathbf{A} \geq 0$ in the positive-semi definite sense, i.e., $\mathbf{A} \geq 0 \Leftrightarrow \mathbf{z}^{\mathrm{T}}\mathbf{A}\mathbf{z} \geq 0$

Efficient Estimator

- From property 3 in the previous slide, it follows for any biased estimator, with bias $\mathbf{b}(\mathbf{x})$:

$$\mathbf{S} = \mathbf{C}_{\mathrm{E}} - \mathbf{b}(\mathbf{x})\mathbf{b}(\mathbf{x})^{\mathrm{T}} - \left(\mathbf{I}_m - \nabla_x^{\mathrm{T}}\mathbf{b}(\mathbf{x})\right)\mathbf{J}^{-1}\left(\mathbf{I}_m - \nabla_x^{\mathrm{T}}\mathbf{b}(\mathbf{x})\right)^{\mathrm{T}} \geq 0 \tag{63}$$

- For an unbiased estimator (i.e., $\mathbf{b}(\mathbf{x}) = \mathbf{0}$), the above leads to the Cramer-Rao bound:

$$\mathbf{S} \geq \mathbf{0} \Leftrightarrow \mathbf{C}_{\mathrm{E}} \geq \mathbf{J}^{-1}. \tag{64}$$

- Efficient estimator is the unbiased estimator for which $\mathbf{C}_{\mathrm{E}} \equiv \mathbf{J}^{-1}$.
- Thus, for an efficient estimator it holds that $\mathbf{S} = \mathbf{0}$; from property 4 in the previous slide, matrix $\mathbf{C}_z$ is of rank m for an efficient estimator.

Efficient Estimator

- For an efficient estimator it holds that $\mathbf{S} = \mathbf{0}$; from property 4 in the previous slide, matrix $\mathbf{C}_z$ is of rank m for an efficient estimator.
- ...the above means that m rows can be written as a linear combination of the other m rows; having in mind the definition of $\mathbf{C}_z$ and the definition of $\mathbf{z}$, the above can be stated as follows:

$$\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y}) = \mathbf{M}\,\nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \Rightarrow \tag{65}$$

$$(\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y}))\nabla_x^{\mathrm{T}} \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) = \mathbf{M}\,\nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\nabla_x^{\mathrm{T}} \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \Rightarrow$$

$$\mathbb{E}_{\mathbf{y}}\left[(\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y}))\nabla_x^{\mathrm{T}} \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right] = \mathbf{M}\,\mathbb{E}_{\mathbf{y}}\left[\nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\nabla_x^{\mathrm{T}} \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x})\right]$$

$$\Rightarrow \mathbf{C}_{12} = \mathbf{M}\,\mathbf{J}$$

$$\Rightarrow \mathbf{M} = \mathbf{C}_{12}\,\mathbf{J}^{-1} = \left(-\mathbf{I}_m + \nabla_x^{\mathrm{T}}\mathbf{b}(\mathbf{x})\right)\mathbf{J}^{-1} = -\mathbf{J}^{-1} \tag{66}$$

$$\Rightarrow \hat{\mathbf{x}}(\mathbf{y}) = \mathbf{x} - \mathbf{M}\,\nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \tag{67}$$

$$\Rightarrow \hat{\mathbf{x}}(\mathbf{y}) = \mathbf{x} + \mathbf{J}^{-1}\,\nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \tag{68}$$

Efficient Estimator Existence

- Thus, an efficient estimator has the following form:

$$\Rightarrow \hat{\mathbf{x}}(\mathbf{y}) = \mathbf{x} + \mathbf{J}^{-1} \nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) \tag{69}$$

- The left-hand side (LHS) is independent of $\mathbf{x}$; thus, an efficient estimator exists iff the right-hand side (RHS) of the above equation is independent of $\mathbf{x}$.
- Notice that $\nabla_x \ln f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) = \mathbf{0} \Rightarrow \hat{\mathbf{x}}(\mathbf{y}) = \mathbf{x}$.
- Remarks follow:
  1. If an efficient exists, it must be a stationary point of the likelihood function; if there is only one such point, it must be the ML estimator.
  2. If the likelihood function has a single maximum and the estimator is efficient, it must be the ML estimator.
  3. ...the above does not mean that all ML estimators are efficient... they may not be!
- ...will see an example at next lecture.

References

[1] Bernard C. Levy, Principles of Signal Detection and Parameter Estimation, Springer 2008.
[2] Instructor notes.

Thank you!

Detection & Estimation Theory: Lectures 17-18

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Outline

- ML Estimates and 1-1 Functions
- Phase Estimation Example
  - Geometric interpretation
- Sufficient Statistic
- UMVU Estimator
- Complete Sufficient Statistic
- The RBLS Theorem (& Proof)

ML Estimates and 1-1 Functions

- Suppose $\hat{\mathbf{x}}_{\mathrm{ML}}(\mathbf{y})$ and $\mathbf{z} = \mathbf{g}(\mathbf{x})$ with $\mathbf{x} = \mathbf{g}^{-1}(\mathbf{z})$, i.e., $\mathbf{g}(\mathbf{x}) = \mathbf{z}$ is a "1-1" mapping.
- Thus, $f_{\mathbf{y}}(\mathbf{y}|\mathbf{z}) = f_{\mathbf{y}}(\mathbf{y}|\underbrace{\mathbf{g}^{-1}(\mathbf{z})}_{\mathbf{x}})$

- Then if $\hat{\mathbf{x}}_{\mathrm{ML}} = \mathbf{g}^{-1}(\hat{\mathbf{z}}) \Rightarrow \hat{\mathbf{z}}_{\mathrm{ML}} = \mathbf{g}(\hat{\mathbf{x}}_{\mathrm{ML}})$.
- However, the transformation does not preserve unbiasedness or efficiency.

Phase estimation example

Now let's examine a phase estimation example:

$$\mathbf{y} = \begin{bmatrix} y_c \\ y_s \end{bmatrix} = A \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \mathbf{v},\ \mathbf{v} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_2),$$

where σ and A are known and θ is unknown.

Thus, $\mathbf{y} \sim N\left(\overbrace{A \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}^{\boldsymbol{\mu}}, \sigma^2 \mathbf{I}_2\right)$.

So,

$$\begin{aligned}
f_{\mathbf{y}}(\mathbf{y}|\theta) &= \frac{1}{\sqrt{(2\pi)^2 \sigma^4}} e^{-\frac{1}{2\sigma^2}\|\mathbf{y}-\boldsymbol{\mu}\|^2} \\
\Rightarrow \ln f_{\mathbf{y}}(\mathbf{y}|\theta) &= -\frac{1}{2} \ln (2\pi)^2 \sigma^4 - \frac{1}{2\sigma^2}\left[(y_c - A\cos\theta)^2 + (y_s - A\sin\theta)^2\right] \\
\Rightarrow L(\mathbf{y}) &\stackrel{\triangle}{=} \ln f_{\mathbf{y}}(\mathbf{y}|\theta) \\
&= -\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}(y_c^2 + y_s^2 + A^2) + \frac{1}{\sigma^2}(A y_c \cos\theta + A y_s \sin\theta)
\end{aligned}$$

Phase estimation example (cont.)

$$\frac{\partial L}{\partial \theta} = \frac{A}{\sigma^2}(-y_c \sin\theta + y_s \cos\theta) = \emptyset$$

$$\Rightarrow y_c \sin\theta = y_s \cos\theta \Rightarrow \tan\theta = \frac{y_s}{y_c}$$

$$\Rightarrow \hat{\theta}_{\mathrm{ML}} = \tan^{-1}\frac{y_s}{y_c}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial \theta^2} &= \frac{A}{\sigma^2}(-y_c \cos\theta - y_s \sin\theta) \\ &= -\frac{A}{\sigma^2}(y_c \cos\theta + y_s \sin\theta)\end{aligned}$$

$$\begin{aligned}d(\theta) = -\mathbb{E}\left[\frac{\partial^2 L}{\partial \theta^2}\right] &= \frac{A}{\sigma^2}\mathbb{E}[y_c \cos\theta + y_s \sin\theta] \\ &= \frac{A}{\sigma^2}\mathbb{E}[A\cos^2\theta + A\sin^2\theta] = \frac{A^2}{\sigma^2}\end{aligned}$$

Phase estimation example (cont.)

$$\mathbb{E}[(\theta - \hat{\theta}(\mathbf{y})^2] \geq \mathrm{J}^{-1}(\theta) = \frac{\sigma^2}{A^2} \simeq \frac{1}{\mathrm{SNR}}$$

- $\mathbb{E}[\hat{\theta}_{\mathrm{ML}}(\mathbf{y})] = ?$
- Is $\hat{\theta}_{\mathrm{ML}}(\mathbf{y})$ efficient?

$$\text{Set} \qquad \left.\begin{aligned} y_c &= r\cos\phi \\ y_s &= r\sin\phi \end{aligned}\right\} \hat{\theta}_{\mathrm{ML}} = \tan^{-1}\frac{y_s}{y_c} = \phi$$

$$\left.\begin{aligned} r > 0, \quad y_c^2 + y_s^2 &= r^2 \\ \tan^{-1}\frac{y_s}{y_c} &= \phi \end{aligned}\right\} \qquad \begin{bmatrix} r \\ \phi \end{bmatrix} \leftarrow \begin{bmatrix} y_c \\ y_s \end{bmatrix}$$

Phase estimation example (cont.)

$$\underset{\begin{bmatrix} y_c \\ y_s \end{bmatrix} \to \begin{bmatrix} r \\ \phi \end{bmatrix}}{\text{Jacobian}} = \begin{bmatrix} \nabla_{\mathbf{y}}^{\intercal} r \\ \nabla_{\mathbf{y}}^{\intercal} \phi \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial y_c} & \frac{\partial r}{\partial y_s} \\ \frac{\partial \phi}{\partial y_c} & \frac{\partial \phi}{\partial y_s} \end{bmatrix} = \begin{bmatrix} \frac{\cancel{2} y_c}{\cancel{2}\sqrt{y_c^2+y_s^2}} & \frac{\cancel{2} y_s}{\cancel{2}\sqrt{y_c^2+y_s^2}} \\ \frac{-y_s \frac{1}{y_c^2}}{1+(\frac{y_s}{y_c})^2} & \frac{\frac{1}{y_c}}{1+(\frac{y_s}{y_c})^2} \end{bmatrix}$$

So,

$$|\det(\text{Jacobian})| = \left| \frac{1}{\sqrt{y_c^2+y_s^2} \cdot \left(1+(\frac{y_s}{y_c})^2\right)} + \frac{\frac{y_s^2}{y_c^2}}{\left(1+\left(\frac{y_s}{y_c}\right)^2\right) \cdot \sqrt{y_c^2+y_s^2}} \right| = \frac{1}{\sqrt{y_c^2+y_s^2}}$$

Phase estimation example (cont.)

$$\text{Hence,} \quad f_{r,\phi}(r,\phi|\theta) = \left.\frac{f_{\mathbf{y}}(\mathbf{y})}{|\text{Jacobian}|}\right|_{\substack{y_c \,=\, r\cos\phi \\ y_s \,=\, r\sin\phi}}$$

$$= \frac{\frac{1}{\sqrt{(2\pi)^2(\sigma^2)^2}}\, e^{-\frac{1}{2\sigma^2}\|\mathbf{y}-\boldsymbol{\mu}\|^2}}{\frac{1}{\sqrt{y_c^2+y_s^2}}}$$

$$= \frac{\sqrt{y_c^2+y_s^2}}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[y_c^2+y_s^2+A^2-2Ay_c\cos\theta-2Ay_s\sin\theta]}$$

$$= \frac{r}{2\pi\sigma^2}\, \underbrace{e^{-\frac{1}{2\sigma^2}[r^2+A^2-2Ar\cos(\phi-\theta)]}}_{\text{even function of } \phi-\theta}$$

$$\text{Thus,} \quad \mathbb{E}[\hat{\theta}_{\text{ML}}] = \mathbb{E}[\phi] = \theta + \cancelto{\emptyset}{\mathbb{E}[\phi-\theta]} = \theta$$

Therefore $\hat{\theta}_{\text{ML}}$ is unbiased.

Phase estimation example (cont.)

Remember that

$$\hat{\mathbf{x}}(\mathbf{y}) = \mathbf{x} + \mathrm{J}^{-1}(\mathbf{x})\nabla_{\mathbf{x}} \ln\left(f_{\mathbf{y}}(\mathbf{y}|\mathbf{x})\right)$$

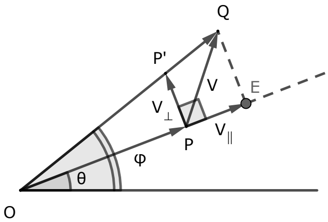
unbiased efficient estimator exists if and only if the RHS does not depend on $\mathbf{x}$.

$$\hat{\mathbf{x}}(\mathbf{y}) = \theta + \frac{\sigma^2}{A^2}\frac{A}{\sigma^2}(-y_c \sin\theta + y_s \cos\theta) = \theta + \frac{r}{A}\sin\left(\phi - \theta\right)$$

As long as $A \simeq r \rightarrow \phi \simeq \theta$ and since $\hat{\theta}_{\mathrm{ML}}(\mathbf{y}) = \phi \simeq \theta$ then,
$\theta + \frac{r}{A}(\phi - \theta) = \phi = \hat{\theta}_{\mathrm{ML}}(\mathbf{y})$
Thus,

$$\mathbb{E}[(\phi - \theta)^2] = \mathbb{E}[(\hat{\theta}_{\mathrm{ML}} - \theta)^2] \simeq \mathrm{J}^{-1} = \frac{\sigma^2}{A^2} = \frac{1}{\mathrm{SNR}}$$

Geometric interpretation



- $|\overrightarrow{\mathrm{OP}}| = A, \ \overrightarrow{\mathrm{OQ}} = \mathbf{y}$
- $\mathbf{V} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_2)$
- $\mathbf{V} = \mathbf{V}_\parallel + \mathbf{V}_\perp$ (orthogonal vector subspace)
- $|\mathbf{V}_\perp|$ is $\mathcal{N}(0, \sigma^2)$

$$\sin(\phi - \theta) \simeq \phi - \theta \simeq \frac{|\mathbf{V}_\perp|}{A} \sim \mathcal{N}(0, \frac{\sigma^2}{A^2})$$

$$\Rightarrow \mathbb{E}[\phi - \theta] = \emptyset \quad \text{and} \quad \mathbb{E}[(\phi - \theta)^2] = \frac{\sigma^2}{A^2}$$

Thus, efficient estimator for high SNR.

Sufficient Statistic

- $\mathbf{y}$ is a sufficient statistic of $\mathbf{x}$ if $f_{\mathbf{y}|\mathbf{s}}$ is independent of $\mathbf{x}$ i.e., all information about $\mathbf{x}$ has been "squeezed" in $f_{\mathbf{s}}(\mathbf{s}|\mathbf{x})$ and there is no leftover information about $\mathbf{x}$ that could be extracted from $f_{\mathbf{y}|\mathbf{s}}$, which means that the latter is independent of $\mathbf{x}$.
- In practice, sufficient statistic $\mathbf{s}(\mathbf{y})$ can be directly found if $f_{\mathbf{y}}(\mathbf{y}|\mathbf{x})$ belongs to the exponential class of densities:

$$f_{\mathbf{y}}(\mathbf{y}|\mathbf{x}) = u(\mathbf{y}) \cdot \exp[\mathbf{x}^{\intercal}\mathbf{s}(\mathbf{y}) - t(\mathbf{t})],$$

  which includes discrete Poisson, Exponential and Gaussian distributions as special cases.

Example 1

- iid $\{y_k\}$'s, $y_k \sim \mathcal{N}(m,u)$, $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \dots & y_{\mathrm{N}} \end{bmatrix}^\mathsf{T}$

$$f_{\mathbf{y}}(m,u) = \prod_{k=1}^{\mathrm{N}} \frac{1}{\sqrt{2\pi u}} \cdot e^{\frac{1}{2u}}(y_k - m)^2$$

$$= \left(\frac{1}{\sqrt{2\pi u}}\right)^{\mathrm{N}} \cdot e^{-\frac{1}{2u}\sum_{k=1}^{\mathrm{N}}(y_k - m)^2}$$

$$= \frac{1}{(2\pi u)^{\frac{\mathrm{N}}{2}}} \cdot e^{-\frac{1}{2u}\left[\sum_{k=1}^{\mathrm{N}} y_k^2 + \mathrm{N}m^2 - 2m\sum_{k=1}^{\mathrm{N}} y_k\right]}$$

$$= \frac{1}{(2\pi u)^{\frac{\mathrm{N}}{2}}} \cdot e^{-\frac{\mathrm{N}m^2}{2u}} \cdot e^{-\frac{1}{2u}\sum_{k=1}^{\mathrm{N}} y_k^2 + \frac{m}{u}\sum_{k=1}^{\mathrm{N}} y_k}$$

Example 1 (cont.)

$$f_{\mathbf{y}}(m,u) = \frac{1}{(2\pi u)^{\frac{\mathrm{N}}{2}}} \cdot e^{-\frac{\mathrm{N}m^2}{2u}} \cdot e^{-\frac{1}{2u}\sum_{k=1}^{\mathrm{N}} y_k^2 + \frac{m}{u}\sum_{k=1}^{\mathrm{N}} y_k}$$

$$= \frac{1}{(2\pi u)^{\frac{\mathrm{N}}{2}}} \cdot e^{-\frac{\mathrm{N}m^2}{2u}} \cdot \exp\left(\underbrace{\begin{bmatrix} \mathrm{N}\frac{m}{u} & -\frac{N}{2u} \end{bmatrix}}_{\mathbf{x}^\intercal} \underbrace{\begin{bmatrix} \frac{1}{\mathrm{N}}\sum_{k=1}^{\mathrm{N}} y_k \\ \frac{1}{\mathrm{N}}\sum_{k=1}^{\mathrm{N}} y_k^2 \end{bmatrix}}_{\mathbf{s}(\mathbf{y})=\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}} \right)$$

i.e. $\mathbf{s}(\mathbf{y})$ is a sufficient statistic for estimating parameter $\mathbf{x}$.

Example 1 (cont.)

Sufficient statistic requires the definition of the unknown parameter

- if u = unknown $\rightarrow$ sufficient statistic is $s_1(\mathbf{y})$
- if m = unknown $\rightarrow$ sufficient statistic is $s_2(\mathbf{y})$

Notice that we have shown that:

$$\hat{m}_{\mathrm{ML}} = \frac{\sum y_k}{\mathrm{N}} = s_1(\mathbf{y})$$

$$\begin{aligned}
\hat{u}_{\mathrm{ML}} = \frac{\sum (y_k - \hat{m}_{\mathrm{ML}})^2}{\mathrm{N}} &= \frac{\sum y_k^2}{\mathrm{N}} + \hat{m}_{\mathrm{ML}}^2 - \frac{2\hat{m}_{\mathrm{ML}}}{\mathrm{N}} \sum y_k \\
&= \frac{\sum y_k^2}{\mathrm{N}} - \hat{m}_{\mathrm{ML}}^2 = s_2(\mathbf{y}) - s_1^2(\mathbf{y})
\end{aligned}$$

Example 2

- iid $\{y_k\}$'s, $y_k \sim$ exponential with known parameter $1/\theta$,
  $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \dots & y_\mathrm{N} \end{bmatrix}^\mathsf{T}$

$$f_{\mathbf{y}|\theta} = \left(\frac{1}{\theta}\right)^\mathrm{N} \cdot e^{-\frac{1}{\theta}\sum y_k} \prod_{k=1}^{\mathrm{N}} u(y_k) \Rightarrow \left| \begin{array}{l} x = \dfrac{1}{\theta} \\ \\ s(\mathbf{y}) = \sum y_k \end{array} \right.$$

$$L(\mathbf{y}|\theta) = \ln\left[f(\mathbf{y}|\theta)\right] = -\mathrm{N}\ln\theta - \frac{s(\mathbf{y})}{\theta},\ y_k \geq \emptyset$$

$$\frac{\partial}{\partial\theta}L(\mathbf{y}|\theta) = -\frac{\mathrm{N}}{\theta} + \frac{s(\mathbf{y})}{\theta^2} = 0 \Rightarrow \hat{\theta}_\mathrm{ML}(\mathbf{y}) = \frac{s(\mathbf{y})}{\mathrm{N}}$$

$$\frac{\partial^2}{\partial\theta^2}L(\mathbf{y}|\theta) = \frac{\mathrm{N}}{\theta^2} - \frac{2s(\mathbf{y})}{\theta^3} = \frac{1}{\theta^2}\left(\mathrm{N} - \frac{2s(\mathbf{y})}{\theta}\right) = \frac{\mathrm{N}}{\theta^2}\left(1 - \frac{2\hat{\theta}}{\theta}\right)$$

Example 2 (cont.)

$$\mathbb{E}\left[\hat{\theta}_{\mathrm{ML}}(\mathbf{y})\right] = \frac{1}{\mathrm{N}}\sum \mathbb{E}[y_k] = \frac{\cancel{\mathrm{N}}\theta}{\cancel{\mathrm{N}}} = \theta, \text{ (unbiased estimate)}$$

$$\mathrm{J}(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}L(\mathbf{y}|\theta)\right] = -\frac{\mathrm{N}}{\theta^2} + \frac{2}{\theta^3}\mathbb{E}[s(\mathbf{y})] = -\frac{\mathrm{N}}{\theta^2} + \frac{2}{\theta^3}\mathrm{N}\theta = \frac{\mathrm{N}}{\theta^2}$$

Thus $\mathbb{E}[(\theta - \hat{\theta}_{\mathrm{ML}})^2] \geq \mathrm{J}^{-1}(\theta) = \frac{\theta^2}{\mathrm{N}}$

$$\begin{aligned}
\mathbb{E}\left[(\theta - \hat{\theta}_{\mathrm{ML}})^2\right] &= \theta^2 + \mathbb{E}\left[\hat{\theta}^2_{\mathrm{ML}}\right] - 2\theta\mathbb{E}\left[\hat{\theta}_{\mathrm{ML}}\right] \\
&= \mathbb{E}\left[\hat{\theta}^2_{\mathrm{ML}}\right] - \theta^2 = \frac{1}{\mathrm{N}^2}\mathbb{E}\left[(\textstyle\sum y_i)^2\right] - \theta^2 \\
&= \frac{1}{\mathrm{N}^2}\left[\mathrm{N}\cdot\mathbb{E}\left[y_i^2\right] + 2\mathbb{E}^2\left[y_i\right]\cdot\binom{\mathrm{N}}{2}\right] - \theta^2
\end{aligned}$$

Example 2 (cont.)

Since $\mathbb{E}[y_i] = \theta = 1/\lambda$
and $\mathbb{E}[y_i^2] - \mathbb{E}^2[y_i] = 1/\lambda^2 = \theta^2 \Rightarrow \mathbb{E}[y_i^2] = 2\theta^2$:

$$\begin{aligned}\mathbb{E}\left[(\theta - \hat{\theta}_{\mathrm{ML}})^2\right] &= \frac{1}{\mathrm{N}^2}\left[\mathrm{N} \cdot 2\theta + 2\theta \cdot \frac{\mathrm{N}(\mathrm{N}-1)}{2}\right] - \theta^2 \\ &= \frac{1}{\mathrm{N}}\left(2\theta^2 + (\mathrm{N}-1)\theta^2\right) - \theta^2 = \frac{\theta^2}{\theta} \equiv \mathrm{J}^{-1}(\theta)\end{aligned}$$

Thus, in this case, the ML unbiased estimate is <u>efficient</u> and $s(\mathbf{y})$ for the parameter θ is $s(\mathbf{y}) = \sum y_k$.

Discussion on unbiased estimates

Set $\mathrm{J}(\hat{\mathbf{x}}, \mathbf{x}) = \mathbb{E}\left[\|\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y})\|_2^2\right] = \mathrm{MSE}$
"Uniform minimum variance <u>unbiased</u> estimate $\hat{\mathbf{x}}_{\mathrm{UMVUE}}(\mathbf{y})$"

$\mathrm{J}(\hat{\mathbf{x}}_{\mathrm{UMVUE}}, \mathbf{x}) \leq \mathrm{J}(\hat{\mathbf{x}}, \mathbf{x})$ for all other unbiased $\hat{\mathbf{x}}$ estimate.

- How do we find it?
  - search for ML estimate, see if it is unbiased and see if it is efficient.
  - if that approach fails, look for <u>complete</u>, <u>sufficient statistic</u>, as well as an unbiased estimator $\check{\mathbf{x}}(\mathbf{y})$.
- Apply RBLS theorem: if $\mathbf{s}(\mathbf{y})$ is <u>complete</u> sufficient statistic, then the estimate $\hat{\mathbf{x}}(\mathbf{s})$ (stemming from RBLS) is a UMVUE of $\mathbf{x}$.

Complete Sufficient Statistic

- What is a <u>complete</u> sufficient statistic?

- Let $\mathbf{s}(\mathbf{y})$ is a sufficient statistic for parameter $\mathbf{x}$.
- $\mathbf{s}$ is complete if $\mathbb{E}\left[\mathbf{h}(\mathbf{s})\right] = \mathbf{0} \Leftrightarrow \mathbf{h}(\mathbf{s}) = \mathbf{0} \Leftrightarrow$ there is at <u>most</u> one unbiased

  estimator of $\mathbf{x}$ depending on $\mathbf{s}$ only.

Note: if $\mathbf{h}(\mathbf{s}) = \mathbf{0} \Rightarrow \mathbb{E}\left[\mathbf{h}(\mathbf{s})\right] = \mathbf{0}$ is trivial.
Obviously, $\mathbb{E}\left[\mathbf{h}(\mathbf{s})\right] = \mathbf{0} \Rightarrow \mathbf{h}(\mathbf{s}) = \mathbf{0}$ is non-trivial

Complete Sufficient Statistic

- How do we find sufficient statistics which are complete? In general it is hard.

- However, for $f_{\mathbf{y}}(\mathbf{y}|\mathbf{x}) = u(\mathbf{y}) \cdot \exp\left[\mathbf{x}^\intercal \mathbf{s}(\mathbf{y}) - t(\mathbf{x})\right]$, $\mathbf{s}(\mathbf{y})$ is complete sufficient statistic!
- ...the above includes Poisson, Exponential and Gaussian.

Rao-Blackwell-Lehmann-Sheffe (RBLS) Theorem

The Rao-Blackwell-Lehmann-Sheffe Theorem states that for an unbiased estimate $\check{\mathbf{x}}(\mathbf{y})$ of $\mathbf{x}$ and a sufficient statistic $\mathbf{s}(\mathbf{y})$, the estimate can be improved:

$$\text{If } \mathbb{E}[\check{\mathbf{x}}(\mathbf{y})] = \mathbf{x}, \tag{1}$$

$$\text{then } \hat{\mathbf{x}}(\mathbf{s}) = \mathbb{E}[\check{\mathbf{x}}(\mathbf{y})|\mathbf{s}] \text{ is unbiased}$$

$$\text{with } \hat{\mathbf{K}}(\mathbf{x}) \leq \check{\mathbf{K}}(\mathbf{x}) \text{ i.e., their differ. is positive semi-definite} \tag{2}$$

$$\text{and } \check{\mathbf{K}}(\mathbf{x}) = \mathbb{E}[(\mathbf{x} - \check{\mathbf{x}})(\mathbf{x} - \check{\mathbf{x}})^\mathsf{T}]$$

$$\hat{\mathbf{K}}(\mathbf{x}) = \mathbb{E}[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^\mathsf{T}]$$

- If $\mathbf{s}$ is complete then $\hat{\mathbf{x}}(\mathbf{s})$ is a uniformly minimum-variance unbiased estimator (UMVUE).

Proof of RBLS Theorem

Proof of RBLS Theorem:
Let's start from the last one and assume that (1), (2) are true:

- If $\hat{\mathbf{s}}$ is complete, then there is at most one unbiased estimate of $\mathbf{x}$ that depends on $\mathbf{s} : \hat{\mathbf{x}}(\mathbf{s})$.
- Suppose that there is a second $\hat{\mathbf{x}}_2(\mathbf{y})$ that achieves smaller $\hat{\mathbf{K}}_2(\mathbf{x}) < \hat{\mathbf{K}}(\mathbf{x})$.
- If we condition on $\mathbf{s}$, then we must get $\hat{\mathbf{x}}(\mathbf{s})$ with $\hat{\mathbf{K}}(\mathbf{x}) \leq \hat{\mathbf{K}}_2(\mathbf{x})$ which is a contradiction.
- Thus, there is no other estimator that minimizes the mean squared error, meaning that $\hat{\mathbf{x}}(\mathbf{s})$ is UMVUE.

Proof of RBLS Theorem (cont.)

Now lets prove that $\mathop{\mathbb{E}}_{\mathbf{s}}[\hat{\mathbf{x}}(\mathbf{s})] = \mathbf{x}$

$$\mathop{\mathbb{E}}_{\mathbf{s}}[\hat{\mathbf{x}}(\mathbf{s})] = \mathop{\mathbb{E}}_{\mathbf{s}}\left[\mathop{\mathbb{E}}_{\mathbf{y}|\mathbf{s}}[\check{\mathbf{x}}(\mathbf{y})|\mathbf{s}]\right] \overset{\triangle}{=} \mathbb{E}[\check{\mathbf{x}}] = \mathbf{x}$$

law of iterated/repeated expectation

Finally we need to prove that $\hat{\mathbf{K}}_{\mathbf{x}} \leq \check{\mathbf{K}}_{\mathbf{x}}$:

$$\begin{aligned}
\check{\mathbf{K}}_{\mathbf{x}} &= \mathbb{E}[(\mathbf{x} - \check{\mathbf{x}})(\mathbf{x} - \check{\mathbf{x}})^{\mathsf{T}}] \\
&= \mathbb{E}[(\underline{\mathbf{x} - \hat{\mathbf{x}}} + \hat{x} - \check{\mathbf{x}})(\underline{\mathbf{x} - \hat{\mathbf{x}}} + \hat{\mathbf{x}} - \check{\mathbf{x}})^{\mathsf{T}}] \\
&= \hat{\mathbf{K}}_{\mathbf{x}} + \cancelto{\emptyset^*}{\mathbb{E}\left[(\mathbf{x} - \hat{\mathbf{x}})\underbrace{(\hat{\mathbf{x}} - \check{\mathbf{x}})}_{\mathbf{g}(\mathbf{y})}{}^{\mathsf{T}}\right]} + \cancelto{\emptyset^*}{\mathbb{E}\left[\underbrace{(\hat{\mathbf{x}} - \check{\mathbf{x}})}_{\mathbf{g}(\mathbf{y})}(\mathbf{x} - \hat{\mathbf{x}})^{\mathsf{T}}\right]} + \\
&\quad + \mathbb{E}[(\hat{\mathbf{x}} - \check{\mathbf{x}})(\hat{\mathbf{x}} - \check{\mathbf{x}})] \qquad (3)
\end{aligned}$$

---

* $\hat{\mathbf{x}} = \mathbb{E}[\check{\mathbf{x}}(\mathbf{y})|\mathbf{s}]$ is the MSE estimate of $\check{\mathbf{x}}(\mathbf{y})$ given $\mathbf{s}$ thus $\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y})$ is orthogonal to any function of $\mathbf{s}$ (or $\mathbf{y}$).

Proof of RBLS Theorem (cont.)

Proof.

Thus from (3) we are left with:

$$\check{\mathbf{K}}_{\mathbf{x}} = \hat{\mathbf{K}}_{\mathbf{x}} + \mathbb{E}[(\hat{\mathbf{x}} - \check{\mathbf{x}})(\hat{\mathbf{x}} - \check{\mathbf{x}})]$$
$$\Rightarrow \check{\mathbf{K}}_{\mathbf{x}} - \hat{\mathbf{K}}_{\mathbf{x}} = \text{covariance matrix}$$
$$\Rightarrow \check{\mathbf{K}}_{\mathbf{x}} - \hat{\mathbf{K}}_{\mathbf{x}} \geq 0 \text{, i.e., } \check{\mathbf{K}}_{\mathbf{x}} - \hat{\mathbf{K}}_{\mathbf{x}} \text{ is positive semi-definite.}$$

□

Example

- iid s $y_k = \begin{bmatrix} y_1 & y_2 & \dots & y_{\mathrm{N}} \end{bmatrix}^{\mathsf{T}}$, $y_k \sim \frac{1}{\theta} \cdot e^{-\frac{y_k}{\theta}}$

$$f(\mathbf{y}|\theta) = \frac{1}{\theta^{\mathrm{N}}} \exp\left(-\frac{s(\mathbf{y})}{\theta}\right) \prod_{k=1}^{\mathrm{N}} u(y_k),\ s(\mathbf{y}) = \sum_{k=1}^{\mathrm{N}} y_k$$

$$\hat{\theta}_{\mathrm{ML}}(\mathbf{y}) = \frac{s(\mathbf{y})}{\mathrm{N}}$$

$$\check{\theta}(\mathbf{y}) = y_1 \text{ since } \mathbb{E}[y_1] = \theta \quad \text{(unbiased estimate)}$$

$$s(\mathbf{y}) = \sum y_k = \underline{\text{complete}} \text{ sufficient statistic}$$

Thus, according to RBLS,

$$\hat{\theta}(s) = \mathop{\mathbb{E}}_{\mathbf{y}}\left[\mathbf{x}(\check{\mathbf{y}})|\mathbf{s}\right] = \mathop{\mathbb{E}}_{\mathbf{y}_1}\left[y_1|\mathbf{s}\right] \quad \text{is an UMVUE}$$

$$= \int y_1 \cdot f_{y_1|s}(y_1|s)\, dy_1$$

Example (cont.)

- Thus, I need to find the $f_{y_1|s}$, $s = \sum_{k=1}^{N} y_k$

$$\text{Set } \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ s \end{bmatrix}}_{\tilde{\mathbf{y}}} = \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}}_{\mathbf{y}}$$

$$\text{Jacobian} = \begin{bmatrix} \nabla_{\mathbf{y}}^{\intercal} y_1 \\ \nabla_{\mathbf{y}}^{\intercal} y_2 \\ \vdots \\ \nabla_{\mathbf{y}}^{\intercal} y_{N-1} \\ \nabla_{\mathbf{y}}^{\intercal} s \end{bmatrix} \overset{*}{=} \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix} = \mathbf{A}$$

---

* $\mathbf{A} \rightarrow$ upper diagonal $\Rightarrow \det(\mathbf{A})$ = product of diagonal elements, and $\det(\mathbf{A}) = 1$

Example (cont.)

$$\nabla_{\mathbf{y}} = \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_{\mathrm{N}}} \end{bmatrix} \quad \text{thus} \quad \begin{aligned} f_{\tilde{\mathbf{y}}}(\tilde{\mathbf{y}}) &\triangleq f_{y_1, y_2, \ldots, y_{\mathrm{N}-1}, s}(\tilde{\mathbf{y}}) \\ &= \left.\frac{f_{\mathbf{y}}(\mathbf{y})}{|\det(\text{Jacobian})|}\right|_{\mathbf{y}=\mathbf{A}^{-1}\cdot\tilde{\mathbf{y}}} \end{aligned}$$

$$s = \sum_{k=1}^{\mathrm{N}-1} y_k + y_n \Rightarrow y_n = s - \sum_{k=1}^{\mathrm{N}-1} y_k$$

$$\Rightarrow f_{y_1, y_2, \ldots, y_{\mathrm{N}-1}, s}(\tilde{\mathbf{y}}) = \frac{1}{\theta^{\mathrm{N}}} \cdot \exp\left(-\frac{s}{\theta}\right) \left(\prod_{k=1}^{\mathrm{N}-1} u(y_k)\right) u\left(s - \sum_{k=1}^{\mathrm{N}-1} y_k\right)$$

Example (cont.)

$$\begin{aligned}
f_{y_1,s} &= \int f_{y_1,y_2,\ldots,y_{\mathrm{N}-1},s}(\tilde{\mathbf{y}})\, dy_2 dy_3 \ldots dy_{\mathrm{N}-1} \\
&= \tfrac{1}{\theta^{\mathrm{N}}} \exp\!\left(-\tfrac{s}{\theta}\right) \int \textstyle\prod_{k=1}^{\mathrm{N}-2} u(y_k) \left[\int u(y_{\mathrm{N}-1}) u\!\left(s - \sum_{k=1}^{\mathrm{N}-1} y_k\right) dy_{\mathrm{N}-1}\right] dy_2 dy_3 \ldots dy_{\mathrm{N}-2} \\
&\overset{*}{=} \tfrac{1}{\theta^{\mathrm{N}}} \exp\!\left(-\tfrac{s}{\theta}\right) \int \textstyle\prod_{k=1}^{\mathrm{N}-2} u(y_k) \cdot u\!\left(s - \sum_{k=1}^{\mathrm{N}-2} y_k\right) \cdot \left(s - \sum_{k=1}^{\mathrm{N}-2} y_k\right) dy_2 dy_3 \ldots dy_{\mathrm{N}-2} \\
&= \cdots = \frac{1}{\theta^{\mathrm{N}}} \exp\left(-\frac{s}{\theta}\right) \frac{(s - y_1)^{\mathrm{N}-2}}{(\mathrm{N}-2)!} u(s - y_1) u(y_1)
\end{aligned}$$

$s(y) : \sum_{k=1}^{\mathrm{N}} y_k =$ sum of N identically distributed exponentials with parameters $\frac{1}{\theta}$ eachs $\Rightarrow s$: Gamma distribution with $f(s) = \Gamma(\mathrm{N}, \theta) = s^{\mathrm{N}-1} \frac{\exp(-s/\theta)}{\Gamma(\mathrm{N}) \cdot \theta^{\mathrm{N}}}$, $\mathbb{E}[s] = \mathrm{N} \cdot \theta$, $\mathrm{Var}(s) = \mathrm{N} \cdot \theta^2$

$^*y_{\mathrm{N}-1} \geq 0,\ s - \sum_{k=1}^{\mathrm{N}-1} y_k \geq 0 \Rightarrow s - \sum_{k=1}^{\mathrm{N}-2} y_k \geq y_{\text{N-1}} \geq 0$

Example (cont.)

- Alternatively, we could integrate $f_{y_1,s}$

$$
\begin{aligned}
\int f_{y_1,s}(y_1,s)\,dy_1 &= \frac{1}{\theta^{\mathrm{N}}}e^{-s/\theta}\frac{1}{(\mathrm{N}-2)!}\int_0^s (s-y_1)^{N-24}\,dy_1 = \\
&= \frac{1}{\theta^{\mathrm{N}}}e^{-s/\theta}\frac{1}{(\mathrm{N}-2)!}\left[-\frac{(s-y_1)^{\mathrm{N}-1}}{\mathrm{N}-1}\right]_0^s \\
&= \frac{1}{\theta^{\mathrm{N}}}e^{-s/\theta}\frac{1}{(\mathrm{N}-1)!}s^{\mathrm{N}-1} \\
&= \frac{1}{\theta^{\mathrm{N}}}e^{-s/\theta}\frac{1}{\Gamma(\mathrm{N})}s^{\mathrm{N}-1}
\end{aligned}
$$

Example (cont.)

- Thus, $f_{y_1|s} = \frac{f_{y_1,s}}{f_s} = \dfrac{\cancel{\frac{1}{\theta^{\mathrm{N}}}}\cancel{e^{-s/\theta}}\frac{(s-y_1)^{\mathrm{N}-2}}{(\mathrm{N}-2)!}u(s-y_1)u(y_1)}{\cancel{\frac{1}{\theta^{\mathrm{N}}}}\frac{1}{(\mathrm{N}-1)!}s^{\mathrm{N}-1}\cancel{e^{-s/\theta}}}$

$$= \frac{(\mathrm{N}-1)}{s^{\mathrm{N}-1}}(s-y_1)^{\mathrm{N}-2}u(s-y_1)u(y_1)$$

- and $\mathbb{E}[y_1|s] = \int y_1 \cdot f_{y_1|s}(y_1|s)\,dy_1$

$$= \int_0^s y_1 \frac{(\mathrm{N}-1)}{s^{\mathrm{N}-1}}(s-y_1)^{\text{N-2}}\,dy_1 = \cdots = \frac{s}{\mathrm{N}}$$

Thus, the RBLS procedure provides $\hat{\theta}(s) = \frac{s}{\mathrm{N}} = \frac{\sum y_k}{\mathrm{N}} \equiv \hat{\theta}_{\mathrm{ML}}$, which can be shown to be both unbiased <u>and</u> efficient!

s complete $\Rightarrow$ at <u>most</u> one unbiased estimate which is a function of s only.

- Trick : find one unbiased estimate which is a function of s only, provided that s is also a complete sufficient statistic. That will be the UMVUE!

Below, we offer an example where transformation offers ML estimate, without preserving unbiased property.

$$\text{if } \hat{\mathbf{x}}(\mathbf{y}) = \hat{\mathbf{x}}_{\text{ML}}(\mathbf{y}),\ \mathbf{z} = \mathbf{g}(\mathbf{x}) \quad \text{1-1} \quad \left[\mathbf{x} = \mathbf{g}^{-1}(\mathbf{z})\right]$$

$$\text{then } \hat{\mathbf{z}}_{\text{ML}}(\mathbf{y}) = \mathbf{g}(\mathbf{x}(\hat{\mathbf{y}})) \quad \text{“ML estimate is parameterization independent”}$$

Example (cont.)

$$\hat{\theta}_{\mathrm{ML}} = \frac{s(\mathbf{y})}{\mathrm{N}} = \frac{\sum y_k}{\mathrm{N}}$$

$$\theta = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{\theta} \Rightarrow \hat{\lambda}_{\mathrm{ML}}(\mathbf{y}) = \frac{1}{\hat{\theta}_{\mathrm{ML}}(\mathbf{y})} = \frac{\mathrm{N}}{s}$$

$$\begin{aligned}
\mathbb{E}\left[\frac{\mathrm{N}}{s}\right] &= \int \frac{\mathrm{N}}{s} f_s(s)\, ds = \frac{\mathrm{N}}{\Gamma(\mathrm{N})\theta^{\mathrm{N}}} \int_{-\infty}^{+\infty} s^{\mathrm{N}-2} e^{-s/\theta} ds \\
&= \frac{\mathrm{N}}{\mathrm{N}-1}\frac{1}{\theta} \int \frac{1}{\theta^{\mathrm{N}-1}} \frac{1}{\Gamma(\mathrm{N}-1)} s^{\mathrm{N}-2} e^{-s/\theta}\, ds \\
&= \frac{\mathrm{N}}{\mathrm{N}-1}\frac{1}{\theta} = \frac{\mathrm{N}}{\mathrm{N}-1}\lambda, \text{ thus the estimator is biased}
\end{aligned}$$

Example (cont.)

- Set $\hat{\lambda}_0 = \frac{\mathrm{N}-1}{\mathrm{N}}\hat{\lambda}_{\mathrm{ML}} = \frac{\mathrm{N}-1}{s}, \quad \mathbb{E}[\hat{\lambda}_0] = \lambda$ (unbiased)
- $\hat{\lambda}_0(s) =$ in a function of s (which is a complete sufficient statistic) only, thus $\hat{\lambda}_0(s) = \hat{\lambda}_{\mathrm{UMVUE}}$
- It can be easily shown that:

$$\left.\begin{array}{l} \text{CRLB: } \lambda^2/\mathrm{N} \\ \mathbb{E}[(\lambda-\lambda_{\mathrm{UMVUE}})^2] = \lambda^2/(\mathrm{N}-2) \\ \mathbb{E}[(x-\hat{\lambda}_{\mathrm{ML}})^2] = \dfrac{\mathrm{N}+2}{(\mathrm{N}-1)(\mathrm{N}-2)}\lambda^2 \end{array}\right\} \begin{array}{l} \mathbb{E}[(\lambda-\lambda_{\mathrm{ML}})^2] > \mathbb{E}[(\lambda-\lambda_{\mathrm{UMVUE}})^2] > \\ \qquad\qquad > \text{CRLB} \end{array}$$

<u>Important Remark</u>: the conditional mean given the <u>complete</u> sufficient statistic should always give the same estimator.

References

Bernard C. Levy, Principles of Signal Detection and Parameter Estimation, Springer 2008.

Thank you!

Detection & Estimation Theory: Lecture 19

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BLUE (Best Linear Unbiased Estimator)

BLUE: Problem Definition

- Suppose that we want to estimate parameter vector $\boldsymbol{\theta}_{(p\times 1)}$ based on measurements $\mathbf{y}_{(N\times 1)}$ with linear estimator:

$$\hat{\boldsymbol{\theta}} = \mathbf{A}_{(p\times N)} \cdot \mathbf{y} \tag{1}$$

- We require unbiased estimator:

$$\mathbb{E}\left[\hat{\boldsymbol{\theta}}\right] = \mathbf{A} \cdot \mathbb{E}\left[\mathbf{y}\right] = \boldsymbol{\theta} \tag{2}$$

that can be achieved if and only if $\mathbb{E}\left[\mathbf{y}\right] = \mathbf{H}_{(N\times p)} \cdot \boldsymbol{\theta}_{(p\times 1)} \Rightarrow$

$$\mathbf{A}_{(p\times N)} \cdot \mathbf{H}_{(N\times p)} = \mathbf{I}_p \tag{3}$$

  - $\mathrm{rank}(\mathbf{I}_p) = p = \min\left(\mathrm{rank}(\mathbf{A}), \mathrm{rank}(\mathbf{H})\right) = \min(N, p)$, thus $N \geq p$

- Set $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \mathbf{a}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_p^{\mathrm{T}} \end{bmatrix}$ (4) and $\mathbf{H} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \cdots & \mathbf{h}_p \end{bmatrix}$ (5)

BLUE Derivation

- from (1) and (4) $\Rightarrow$
$$\hat{\theta}_i = \mathbf{a}_i^{\mathrm{T}} \cdot \mathbf{y} \qquad (6)$$
- from (4), (5) and (3) $\Rightarrow \mathbf{a}_i^{\mathrm{T}} \mathbf{h}_j = \delta_{ij}$
- $\mathrm{var}(\hat{\theta}_i) = \mathbb{E}\left[\left(\hat{\theta}_i - \mathbb{E}[\hat{\theta}_i]\right)^2\right] = \mathbb{E}\left[\left(\mathbf{a}_i^{\mathrm{T}} \mathbf{y} - \mathbf{a}_i^{\mathrm{T}} \mathbb{E}[\mathbf{y}]\right)^2\right]$

$$= \mathbb{E}\left[\left[\mathbf{a}_i^{\mathrm{T}} (\mathbf{y} - \mathbb{E}[\mathbf{y}])\right]^2\right]$$

$$= \mathbb{E}\left[\mathbf{a}_i^{\mathrm{T}} (\mathbf{y} - \mathbb{E}[\mathbf{y}]) (\mathbf{y} - \mathbb{E}[\mathbf{y}])^{\mathrm{T}} \mathbf{a}_i\right] = \mathbf{a}_i^{\mathrm{T}} \mathbf{K}_{\mathbf{y}} \ \mathbf{a}_i$$

$$\Rightarrow \mathrm{var}(\hat{\theta}_i) = \mathbf{a}_i^{\mathrm{T}} \mathbf{K}_{\mathbf{y}} \ \mathbf{a}_i \qquad (7)$$

BLUE Derivation

- Minimize $\mathrm{var}(\hat{\theta}_i) = \mathbf{a}_i^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}\ \mathbf{a}_i$, for $i = 1, 2, \cdots, p$ subject to the constraints $\mathbf{a}_i^{\mathrm{T}} \mathbf{h}_j = \delta_{ij},\ i, j \in \{1, 2, \cdots, p\}$
- We have p constraints for each $\mathbf{a}_i$. Since each $\mathbf{a}_i$ is free to assume any value, independently of the others, we actually have p separate minimization problems linked only by the constraints:

$$J_i = \mathbf{a}_i^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}\ \mathbf{a}_i + \sum_{j=1}^{p} \left(\lambda_j^{(i)} (\mathbf{a}_i^{\mathrm{T}} \mathbf{h}_j - \delta_{ij}) \right), \boldsymbol{\lambda}_i = \left[\lambda_1^{(i)}\ \lambda_2^{(i)} \cdots\ \lambda_p^{(i)} \right]^{\mathrm{T}}$$

$$\frac{\partial J_i}{\partial \mathbf{a}_i} = 2\mathbf{K}_{\mathbf{y}} \mathbf{a}_i + \sum_{j=1}^{p} \lambda_j^{(i)} \mathbf{h}_j = 2\mathbf{K}_{\mathbf{y}} \mathbf{a}_i + \mathbf{H} \boldsymbol{\lambda}_i = \mathbf{0}$$

$$\Rightarrow \frac{\partial J_i}{\partial \mathbf{a}_i} = \mathbf{0} \Rightarrow \mathbf{a}_i = -\frac{1}{2} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \boldsymbol{\lambda}_i \tag{8}$$

BLUE Derivation

- To find $\boldsymbol{\lambda}_i$, we need to exploit the constraints:

$$\mathbf{a}_i^{\mathrm{T}} \cdot \mathbf{h}_j = \mathbf{h}_j^{\mathrm{T}} \mathbf{a}_i = \delta_{ij}, \qquad j = 1, 2, \cdots, p$$

  where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

- from (5) and the above $\Rightarrow \mathbf{H}^{\mathrm{T}} \cdot \mathbf{a}_i = \mathbf{e}_i$, which is a vector with all zeros apart from position i, where it is one:

$$\mathbf{H}^{\mathrm{T}} \cdot \mathbf{a}_i = \mathbf{e}_i \Rightarrow \mathbf{H}^{\mathrm{T}} \cdot \left(-\frac{1}{2} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \boldsymbol{\lambda}_i \right) = \mathbf{e}_i$$

$$\Rightarrow -\frac{1}{2} \boldsymbol{\lambda}_i = \underbrace{\left(\mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \right)^{-1}}_{\text{assuming invertibility of } \mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H}} \tag{9}$$

- from (8) and (9) $\Rightarrow$

$$\mathbf{a}_{i_{opt}} = \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \left(\mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \right)^{-1} \mathbf{e}_i$$

BLUE Derivation

$$\hat{\boldsymbol{\theta}} = \mathbf{A} \cdot \mathbf{y} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \mathbf{a}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_p^{\mathrm{T}} \end{bmatrix} \cdot \mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{e}_1^{\mathrm{T}} \\ \mathbf{e}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{e}_p^{\mathrm{T}} \end{bmatrix}}_{\mathbf{I}_p} \cdot \left(\mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \right)^{-1} \mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \cdot \mathbf{y}$$

$$\Rightarrow \hat{\boldsymbol{\theta}} = \left(\mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \right)^{-1} \mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \cdot \mathbf{y}$$

Example 1

- $\mathbf{y} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$, $\mathbf{w}$ has zero mean $\mathbb{E}[\mathbf{w}] = \mathbf{0}$ and $\mathbb{E}[\mathbf{w}\mathbf{w}^{\mathrm{T}}] = \mathbf{K}_{\mathbf{y}}$, so $\mathbb{E}[\mathbf{y}] = \mathbf{H}\boldsymbol{\theta}$
- $$\begin{aligned}\hat{\boldsymbol{\theta}} - \mathbb{E}\left[\hat{\boldsymbol{\theta}}\right] &= \left(\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{y} - \mathbb{E}\left[\hat{\boldsymbol{\theta}}\right] \\ &= \cancel{\left(\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{H}\boldsymbol{\theta}} - \cancelto{\boldsymbol{\theta}}{\mathbb{E}\left[\hat{\boldsymbol{\theta}}\right]} + \\ &\quad + \left(\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{w} \\ &= \left(\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{w}\end{aligned}$$
- Thus,

$$\begin{aligned}&\mathbb{E}\left[\left(\hat{\boldsymbol{\theta}} - \mathbb{E}\left[\hat{\boldsymbol{\theta}}\right]\right)\left(\hat{\boldsymbol{\theta}} - \mathbb{E}\left[\hat{\boldsymbol{\theta}}\right]\right)^{\mathrm{T}}\right] = \\ &= \left(\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\underbrace{\mathbf{K}_{\mathbf{y}}\mathbf{K}_{\mathbf{y}}^{-1}}_{\mathbf{I}}\mathbf{H}\left(\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{H}\right)^{-1} \\ &= \left(\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{H}\right)^{-1} \equiv \mathbf{C}_{\hat{\boldsymbol{\theta}}}\end{aligned}$$

$$\Rightarrow \mathrm{var}(\hat{\theta}_i) = \left[\left(\mathbf{H}^{\mathrm{T}}\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{H}\right)^{-1}\right]_{ii} \tag{10}$$

Example 1

- This could be also seen from

$$\hat{\theta}_i = \mathbf{a}_{i_{opt}}^{\mathrm{T}} \cdot \mathbf{y} = \mathbf{e}_i^{\mathrm{T}} \left(\mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \right)^{-1} \mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \cdot \mathbf{y}$$

and

$$\begin{aligned}
\mathrm{var}\hat{\theta}_i &= \mathbf{a}_{i_{opt}}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}} \; \mathbf{a}_{i_{opt}} \\
&= \mathbf{e}_i^{\mathrm{T}} \left(\mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \right)^{-1} \mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{K}_{\mathbf{y}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \left(\mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \right)^{-1} \mathbf{e}_i \\
&= \mathbf{e}_i^{\mathrm{T}} \left(\mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \right)^{-1} \mathbf{e}_i \\
&\overset{(7)}{=} \left[\left(\mathbf{H}^{\mathrm{T}} \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{H} \right)^{-1} \right]_{ii}
\end{aligned}$$

Remarks

- *Remark 1:* MVUE for linear Gaussian Case $\equiv$ BLUE, if $\mathbf{y} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}, \mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$, then $\hat{\boldsymbol{\theta}} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{y} \equiv$ BLUE is also UMVUE
- *Remark 2 (Gauss-Markov Theorem):* if the data are of the general linear model form $\mathbf{y} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$, where $\mathbf{H}$ is a <u>known</u> $N \times p$ matrix, $\boldsymbol{\theta}$ is a $p \times 1$ vector of parameters to be estimated, and $\mathbf{w}$ is a $N \times 1$ noise vector with zero mean and covariance $\mathbf{C}$[1], then BLUE of $\boldsymbol{\theta}$ is

$$\hat{\boldsymbol{\theta}} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{y} \quad \text{and} \quad \mathbf{C}_{\hat{\boldsymbol{\theta}}} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{H}\right)^{-1}$$

[1] N should be greater or equal than p ($N \geq p$)

Example 2

- $y[n] = A + w[n],\ n = 0, 1, \cdots, N-1 : w[n]$ white noise with variance σ^2 (not necessarily Gaussian)

$$\mathbb{E}\left[w[n]\right] = 0 \quad \text{and} \quad \mathbb{E}\left[w[n]w[n+m]\right] = \sigma^2 \delta[m]$$

$$\mathbf{y} = \underbrace{\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{H}_{N\times 1}} \cdot A = \underbrace{\begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[N-1] \end{bmatrix}}_{\mathbf{w}}$$

$$\mathbb{E}[\mathbf{y}] = \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{H}} \cdot A, \quad \mathbb{E}[\mathbf{w}] = 0 \quad \text{and} \quad \mathbb{E}[\mathbf{w}\mathbf{w}^{\mathrm{T}}] = \sigma^2 \mathbf{I}_N$$

Example 2

- Thus,

$$\hat{\boldsymbol{\theta}} = \left(\frac{N}{\sigma^2}\right)^{-1} \cdot \frac{1}{\sigma^2} \sum_{n=0}^{N-1} y[n]$$

$$= \frac{1}{N} \sum_{n=1}^{N-1} y[n] = \hat{A}$$

$$\mathbb{E}\left[A - \hat{A}\right] = \left(\frac{1}{\sigma^2} N\right)^{-1} = \frac{\sigma^2}{N}$$

Example 3

- We had seen that for y_k i.i.d. $\sim \underbrace{\mathcal{N}(\cancel{m}^{0}, u)}_{m\ known} = \mathcal{N}(0, u)$

$$\hat{u}_{\text{ML}} = \frac{1}{N}\sum_{k=1}^{N} y_k^2 \quad \text{and} \quad \text{CRLB} = \frac{2u^2}{N}$$

- Is $\hat{u}_{\text{ML}}$ efficient?
- What is the BLUE estimate of u?

Example 3

- $\mathbb{E}[\hat{u}_{\mathrm{ML}}] = \frac{1}{N} N u = u$ unbiased
- $$\mathbb{E}\left[(\hat{u}_{\mathrm{ML}} - u)^2\right] = \mathbb{E}\left[\hat{u}_{\mathrm{ML}}^2\right] - u^2 = \frac{1}{N^2}\left(\sum_{k=1}^{N} y_k^2\right)^2 - u^2$$
$$= \frac{1}{N^2}\left(N\mathbb{E}[y_k^4] + \binom{N}{2} 2u^2\right) - u^2$$
$$\overset{2}{=} \frac{1}{N^2}\left(N \cdot 3u^2 + \frac{N!}{\cancel{2}!(N-2)!}\cancel{2}u^2\right) - u^2$$
$$= \frac{3u^2}{N} + \frac{N-1}{N}u^2 - u^2 = \frac{2u^2}{N} \equiv \mathrm{CRLB}$$
- Thus, $\hat{u}_{\mathrm{ML}}(\mathbf{y}) = \frac{1}{N}\sum_{k=1}^{N} y_k^2$ is efficient.

---

[2] $y \sim \mathcal{N}(0, u), \sigma_u = \sqrt{u} \Rightarrow$

$$\mathbb{E}[y^n] = \begin{cases} (\sigma_u)^n \cdot 1 \cdot 3 \cdots (n-1), & \text{n even,} \\ \emptyset, & \text{n odd,} \end{cases}$$

Example 3 - BLUE estimate

- Search for BLUE of u:

$$\hat{u}_{\text{BLUE}}(\mathbf{y}) = \mathbf{a}^{\mathrm{T}}\mathbf{y} = \sum_{k=1}^{N} a_k y_k$$

$$\mathbb{E}[\hat{u}_{\text{BLUE}}] = \sum a_k \cancelto{\emptyset}{\mathbb{E}[y_k]} = \emptyset \neq u \quad \text{Biased}$$

- However, we can use $z_k = y_k^2$ (data transformation) and test BLUE on the transformed data:[3]

$$\hat{u}_{\text{BLUE}}(\mathbf{z}) = \sum a_k z_k = \sum_{k=1}^{N} a_k y_k^2 \Rightarrow \tag{11}$$

$$\mathbb{E}[\hat{u}_{\text{BLUE}}] = \underbrace{\sum_{k=1}^{N} a_k u}_{\sum_{k=1}^{N} a_k = 1} = u \tag{12}$$

[3] z_k iid, $\sum a_k = 1 \Rightarrow a_k = \frac{1}{N}$

References

Bernard C. Levy, Principles of Signal Detection and Parameter Estimation, Springer 2008.

Thank you!

Detection & Estimation Theory: Lectures 20-21

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Outline

- Introduction to Composite Hypothesis Testing and UMP/GLRT
- GLRT: Examples and Properties
- Asymptotic Optimality of the GLRT

Composite Hypothesis Testing

- Composite Hypothesis Testing: Problem of both Detection and Estimation!
- Problem definition:
  1. $\mathcal{H}_0 : \mathbf{y} \sim f_{\mathbf{y}}\left(\mathbf{y}|\mathbf{x}, \mathcal{H}_0\right)$, $\mathbf{x} \in \mathcal{X}_0$
     $\mathcal{H}_1 : \mathbf{y} \sim f_{\mathbf{y}}\left(\mathbf{y}|\mathbf{x}, \mathcal{H}_1\right)$, $\mathbf{x} \in \mathcal{X}_1$
  2. $\mathbf{x}$ defines $\mathcal{H}_j$; if $\mathcal{X}_0 \equiv \mathcal{X}_1$ there would be no way to distinguish between $\mathcal{H}_0, \mathcal{H}_1 \Rightarrow$ no way to estimate $\mathbf{x}$
- Example: $y[k] = As[k] + v[k]$, $k \in \{1, \ldots, N\}$
  1. A unknown, v WGN with $v \sim \mathcal{N}\left(0, \sigma^2\right)$, σ^2 unknown
  2. $\mathbf{y} = \begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[N] \end{bmatrix} = A \begin{bmatrix} s[1] \\ s[2] \\ \vdots \\ s[N] \end{bmatrix} + \begin{bmatrix} v[1] \\ v[2] \\ \vdots \\ v[N] \end{bmatrix}$
  3. $\mathcal{H}_0$: $A = 0$, σ^2 unknown $\Rightarrow \mathcal{X}_0 = \left\{\left(A, \sigma^2\right) : A = 0\right\}$
     $\mathcal{H}_1$: $A \neq 0$, σ^2 unknown $\Rightarrow \mathcal{X}_1 = \left\{\left(A, \sigma^2\right) : A \neq 0\right\}$
  4. σ^2 is common to both hypothesis, thus σ^2 does not play a role in determining which hypothesis holds (i.e., it is a "nuisance parameter").

Composite Hypothesis Testing

- Remark: In the above example $\mathcal{X}_0 \cap \mathcal{X}_1 = \emptyset$, even though σ^2 is common - remember $\mathcal{X} = \left(A, \sigma^2\right)$. If under $\mathcal{X}_0$, $\mathcal{X}_1$ the observation distribution is the same, then the detection problem cannot be solved, unless $\mathcal{X}_0 \cup \mathcal{X}_1 = \emptyset$.
- Special Case: $\mathcal{H}_1$ is composite but $\mathcal{H}_0$ is "simple". This means that the domain $\mathcal{X}_0$ reduces to a single point $\mathbf{x}_0 \rightarrow$ easier analysis than the case where both hypotheses are composite.
- Example: The previous example with σ^2 known instead
  1. $\mathcal{H}_0$: $\mathcal{X}_0$: $\left(A, \sigma^2\right)$: $A = 0$ and σ^2 fixed and known)
  2. set $\mathcal{Y} = \mathcal{Y}_0 \cup \mathcal{Y}_1$ $(\mathcal{Y}_0 \cup \mathcal{Y}_1 = \emptyset)$
  3. $\mathcal{Y}_j = \mathbf{y} : \delta\left(\mathbf{y}\right) = j$, $j \in 0, 1$, $\delta\left(\cdot\right)$ decision rule
  4. Probability of detection:[1]

$$\mathrm{P_D}\left(\delta, \mathbf{x}\right) = \Pr\left(\delta = 1 | \mathbf{x}, \mathcal{H}_1\right) = \int_{\mathcal{Y}_1} f\left(\mathbf{y} | \mathbf{x}, \mathcal{H}_1\right) d\mathbf{y}$$

  5. Probability of false alarm:[1]

$$\mathrm{P_F}\left(\delta, \mathbf{x}\right) = \Pr\left(\delta = 1 | \mathbf{x}, \mathcal{H}_0\right) = \int_{\mathcal{Y}_1} f\left(\mathbf{y} | \mathbf{x}, \mathcal{H}_0\right) d\mathbf{y}$$

[1] function of $\mathbf{x}$

Composite Hypothesis Testing

- When $\mathrm{P_D}(\delta, \mathbf{x})$ is viewed as a function of $\mathbf{x}$, it is called the "power of the test"
- Neyman–Pearson approach is followed (even though Bayesian approach is also possible):
  1. set upper bound for probability of false alarm
  2. $\max_{x \in \mathcal{X}_0} \mathrm{P_F}(S, \mathbf{x}) \geq a$ (1)
  3. a is called the size of the test
  4. Then, among all tests δ obeying Eq. (1), we say that δ_{UMP} is a uniformly most powerful (UMP) test if it satisfies:

$$\mathrm{P_D}(\delta, \mathbf{x}) \leq \mathrm{P_D}(\delta_{\mathrm{UMP}}, \mathbf{x})$$

     for all $\mathbf{x} \in \mathcal{X}_1$.

     I. very strong property - rarely we find UMP
     II. UMP test δ_{UMP} cannot depend on $\mathbf{x}$
     III. if $\mathbf{x}$ is viewed as being fixed, δ_{UMP} must be the optimum test in the sense of Neyman-Pearson tests (max $\mathrm{P_D}$ for bounded $\mathrm{P_F}$), so it must take the form of a LRT, possibly involving randomization.

Composite Hypothesis Testing - UMP

- Thus, from II) and III) we need to find LRT and then try to transform it in such a way that the parameter vector $\mathbf{x}$ disappears from the test statistic. Then the threshold of the test is computed in such a way that the $\mathrm{P_F}$ upper bound is satisfied. If that is possible, then a UMP test exists!
- Example: The previous example rewritten:
  - $\mathcal{H}_1$: $y[k] = As[k] + w[k]$, $A > 0$ unknown, $w[k] \sim \mathcal{N}\left(0, \sigma^2\right)$ (WGN)
  - $\mathcal{H}_0 : y[k] = w[k]$, $A \neq 0$
  - $k \in 1, 2, \ldots, N$
  - Thus,
    - $\mathcal{H}_1$: $\mathbf{y} = A\mathbf{s} + \mathbf{w}$, $A > 0$, $\mathbf{w} \sim \mathcal{N}\left(0, \sigma^2 I_N\right)$
    - $\mathcal{H}_0$: $\mathbf{y} = \mathbf{w}$
  - case I: σ^2 known
  - $\mathcal{X}_1 = \{A > 0\}$ (or $A < 0$)(one-sided test), $\mathcal{X}_0 = \{A = 0\}$ "simple"

Composite Hypothesis Testing - UMP

- LRT: $L(\mathbf{y}|A) = \frac{f(\mathbf{y}|A>0,\mathcal{H}_1)}{f(\mathbf{y}|A=0,\mathcal{H}_0)} = \frac{f(\mathbf{y}|A>0)}{f(\mathbf{y}|A=0)} \overset{\mathcal{H}_1}{\geq} \tau$[1]
- $f(\mathbf{y}|A>0) = \frac{1}{\sqrt{(2\pi)^N(\sigma^2)^N}} e^{-\frac{1}{2\sigma^2}||\mathbf{y}-A\mathbf{s}||^2}$
- $$\begin{aligned} ||\mathbf{y}-A\mathbf{s}||^2 &= (\mathbf{y}-A\mathbf{s})^{\mathrm{T}}(\mathbf{y}-A\mathbf{s}) \\ &= \left(\mathbf{y}^{\mathrm{T}}-A\mathbf{s}^{\mathrm{T}}\right)(\mathbf{y}-A\mathbf{s}) \\ &= ||\mathbf{y}||^2 - A\mathbf{y}^{\mathrm{T}}\mathbf{s} - A\mathbf{s}^{\mathrm{T}}\mathbf{y} + A^2||\mathbf{s}||^2 \\ &= ||\mathbf{y}||^2 - 2A\mathbf{s}^{\mathrm{T}}\mathbf{y} + A^2||\mathbf{s}||^2 \end{aligned}$$
- set $||\mathbf{s}||^2 = \mathbf{s}^{\mathrm{T}}\mathbf{s} = \sum_{k=1}^{N} s^2[k] = E$
- $$L(\mathbf{y}|A) = e^{-\frac{1}{2\sigma^2}\left(-2A\mathbf{s}^{\mathrm{T}}\mathbf{y}+A^2||\mathbf{s}||^2\right)} \Rightarrow$$
  $$\ln L(\mathbf{y}|A) = \frac{A}{\sigma^2}\mathbf{s}^{\mathrm{T}}\mathbf{y} - \frac{A^2}{2\sigma^2}E \Rightarrow$$
  $$\frac{A}{\sigma^2}\mathbf{s}^{\mathrm{T}}\mathbf{y} - \frac{A^2E}{2\sigma^2} \overset{\mathcal{H}_1}{\geq} \ln\tau$$

[1] yet to be specified

Composite Hypothesis Testing - UMP

- Remember that we need a test, which does not depend on the unknown parameter(s). Thus, we need to get rid of A.
- $$A > 0 \Rightarrow \frac{1}{\sigma^2}\mathbf{s}^{\mathrm{T}}\mathbf{y} - \frac{AE}{2\sigma^2} \overset{\mathcal{H}_1}{\geq} \frac{1}{A}\ln\tau \Rightarrow$$
  $$\frac{\mathbf{s}^{\mathrm{T}}\mathbf{y}}{\sqrt{E}} - \frac{A\sqrt{E}}{2} \overset{\mathcal{H}_1}{\geq} \frac{\sigma^2}{A\sqrt{E}}\ln\tau \Rightarrow$$
  $$\frac{\mathbf{s}^{\mathrm{T}}\mathbf{y}}{\sqrt{E}} \overset{\mathcal{H}_1}{\geq} \frac{A\sqrt{E}}{2} + \frac{\sigma^2}{A\sqrt{E}}\ln\tau \overset{\triangle}{=} \eta$$
- $s(\mathbf{y}) = \frac{\mathbf{s}^{\mathrm{T}}\mathbf{y}}{\sqrt{E}}$ is Gaussian
  - $\mathbb{E}[s(\mathbf{y})] = 0$ under $\mathcal{H}_0$ and
    $\mathbb{E}[s(\mathbf{y})] = \frac{\mathbf{s}^{\mathrm{T}}A\mathbf{s}}{\sqrt{E}} = \frac{A\|\mathbf{s}\|}{\sqrt{E}} = \frac{AE}{\sqrt{E}} = A\sqrt{E}$ under $\mathcal{H}_1$
  - $\mathrm{Var}(\mathbf{s}(\mathbf{y})) = \mathbb{E}[\frac{1}{E}\mathbf{s}^{\mathrm{T}}\mathbf{y}\mathbf{y}^{\mathrm{T}}\mathbf{s}] = \frac{1}{E}[\mathbf{s}^{\mathrm{T}}\sigma^2 I_N\mathbf{s}] = \frac{E}{E}\sigma^2 = \sigma^2$
  - Thus $s(\mathbf{y}) \sim \mathcal{N}\left(0, \sigma^2\right)$ under $\mathcal{H}_0$ and $s(\mathbf{y}) \sim \mathcal{N}\left(A\sqrt{E}, \sigma^2\right)$ under $\mathcal{H}_1$

Composite Hypothesis Testing - UMP

- $s(\mathbf{y})$ independent of A, need to calculate η.
- $$\begin{aligned} \mathrm{P_F}(\delta = 1 | \mathcal{H}_0, A = 0) &= \Pr(s(\mathbf{y}) \geq \eta | \mathcal{H}_0, A = 0) \\ &= \int_y^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} s^2} ds \\ &= \int_{\frac{\eta}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} t^2} dt \\ &= Q\left(\frac{\eta}{\sigma}\right) \end{aligned}$$

  where $t = \frac{s}{\sigma} \Rightarrow dt = \frac{1}{\sigma} ds$
- Be careful: we don't need $\max_{\mathbf{x} \in \mathcal{X}_0} P_F$ since $\mathbf{x} = \mathbf{x}_0$ (simple)
- $Q\left(\frac{\eta}{\sigma}\right) = a \Rightarrow \frac{\eta}{\sigma} = Q^{-1}(a) \Rightarrow \eta = \sigma \cdot Q^{-1}(a)$
- Thus the test $s(\mathbf{y}) \overset{\mathcal{H}_1}{\geq} \eta$ is UMP!

Composite Hypothesis Testing - UMP

- The power of the test can be calculated as follows:
  $$\mathrm{P_D}(A) = \Pr\left(s(\mathbf{y}) \geq \eta | A > 0, \mathcal{H}_1\right) \quad = \int_{y}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(s - A\sqrt{E}\right)^2} ds$$
- set $\frac{s-A\sqrt{E}}{\sigma} = t$
- $$\begin{aligned} \mathrm{P_D}(A) &= Q\left(\frac{y - A\sqrt{E}}{\sigma}\right) \\ &= 1 - Q\left(\frac{A\sqrt{E} - \eta}{\sigma}\right) \\ &= 1 - Q\left(\frac{A\sqrt{E}}{\sigma} - Q^{-1}(a)\right) \end{aligned}$$
  which is monotone increasing with A.
- Remark: We managed to find UMP since we managed to get rid of the dependence of $s(\mathbf{y})$ from A. The same would be possible for $A < 0$. But it could be impossible for $A \neq 0$, since the order of the inequality would be unknown. Thus, UMP test exists only for the one-sided test $A > 0$ (or $A < 0$).

Composite Hypothesis Testing - UMP

- case II: σ^2 unknown
- In that case hypothesis $\mathcal{H}_0$ is also composite since $\left(A, \sigma^2\right) = \left(0, \sigma^2\right)$ (σ^2 unknown)
- The LRT derivation still stands. How do we select η?
- one approach: set $\eta = +\infty \Rightarrow \mathrm{P_D} = 0$ (not very good)
- second approach: $\sigma_L^2 \leq \sigma^2 \leq \sigma_U^2$ given that $\mathrm{P_F} = Q\left(\frac{\eta}{\sigma}\right) = a$
  $\mathrm{P_F} = Q\left(\frac{\eta}{\sigma}\right) \leq Q\left(\frac{\eta}{\sigma_U}\right) = a$, since $Q\left(x\right)$ is increasing with decreasing x.
- Thus $\eta = \sigma_U \cdot Q^{-1}\left(a\right)$ satisfies $\mathrm{P_F} \leq a$ and thus UMP still exists!

Composite Hypothesis Testing - GLRT

- UMP is rarely found. We revert to generalised likelihood ratio test (GLRT).
- GLRT: Suboptimal technique in general, even though it can provide UMP tests in special cases.
- $\mathcal{H}_0$: $\mathbf{y} \sim f(\mathbf{y}|\mathbf{x}, \mathcal{H}_0)$, $\mathbf{x} \in \mathcal{X}_0$
- $\mathcal{H}_1$: $\mathbf{y} \sim f(\mathbf{y}|\mathbf{x}, \mathcal{H}_1)$, $\mathbf{x} \in \mathcal{X}_1$
- $L_G(\mathbf{y}) = \frac{\max_{\mathbf{x}\in\mathcal{X}_1} f(\mathbf{y}|\mathbf{x},\mathcal{H}_1)}{\max_{\mathbf{x}\in\mathcal{X}_0} f(\mathbf{y}|\mathbf{x},\mathcal{H}_0)} = \frac{f(\mathbf{y}|\hat{\mathbf{x}}_1,\mathcal{H}_1)}{f(\mathbf{y}|\hat{\mathbf{x}}_0,\mathcal{H}_0)}$,
  $\hat{\mathbf{x}}_i = \arg\max_{\mathbf{x}_i\in\mathcal{X}_i} f(\mathbf{y}|\mathbf{x},\mathcal{H}_i)$
- GLR is obtained by replacing the unknown parameter vector $\mathbf{x}$ by its estimate!
- $L_G(\mathbf{y}) \overset{\mathcal{H}_1}{\geq} \tau$ and then we select τ by the size of the test:

$$\max_{\mathbf{x}\in\mathcal{X}} Pr\left(L_G(\mathbf{y}) \geq \tau | \mathcal{H}_0, \mathbf{x}\right) \leq a,$$

  where a is the size of the test.

Summary - Composite Hypothesis Testing with 3 approaches:

- UMP(τ) (to be explained later)
- GLRT
- "Frequentist approach": treat $\mathbf{x}$ as a random vector rather than a constant (i.e. non-random) parameter:
  1. $f\left(\mathbf{y}|\mathcal{H}_i\right) = \int f\left(\mathbf{y}|\mathbf{x}, \mathcal{H}_i\right) f\left(\mathbf{x}|\mathcal{H}_i\right) d\mathbf{x}$
  2. Notice that $f\left(\mathbf{y}|\mathcal{H}_i\right) = \mathbb{E}_{\mathbf{x}|\mathcal{H}_i}\left[f\left(\mathbf{y}|\mathbf{x}, \mathcal{H}_i\right)\right]$ and $L = \frac{\mathbb{E}_{\mathbf{x}|\mathcal{H}_1}[f(\mathbf{y}|\mathbf{x}, \mathcal{H}_1)]}{\mathbb{E}_{\mathbf{x}|\mathcal{H}_0}[f(\mathbf{y}|\mathbf{x}, \mathcal{H}_0)]}$
  3. In other words, set $f\left(\mathbf{x}|\mathcal{H}_i\right)$, calculate the above and then use detection theory.

Example

- $\mathbf{y} = \begin{bmatrix} y_c \\ y_s \end{bmatrix} = A \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \mathbf{v}$, $\mathbf{v} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2 \mathbf{I}_2\right)$
- Incoherent Detection
  - σ^2 known, A, θ unknown
  - $\mathcal{H}_0$: $A = 0$
    $\mathcal{H}_1$: $A \neq 0$
- Polar Coordinates:
  1. $y_c = r\cos\phi$
  2. $y_s = r\sin\phi$
  3. $r = \sqrt{y_c^2 + y_s^2}$
  4. $\phi = \tan^{-1} \frac{y_s}{y_c}$
- $f(r, \phi | A, \theta) = \frac{r}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left(A^2 + r^2 - 2Ar\cos(\phi - \theta)\right)}$
  (we have showed this in a previous lecture).

Example

- $\mathcal{H}_1$: $\mathbf{y} = A \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \mathbf{v}$
- $$f(\mathbf{y}|A,\theta) = \mathcal{N}\left(A \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \sigma^2 \mathbf{I}_2\right)$$
$$= \frac{1}{\sqrt{(2\pi)^2 \sigma^4}} e^{-\frac{1}{2\sigma^2}\left[(y_c - A\cos\theta)^2 + (y_s - A\sin\theta)^2\right]}$$
$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left(y_c^2 + y_s^2 + A^2 - 2Ay_c\cos\theta - 2Ay_s\sin\theta\right)}$$
- $$\ln[f(\mathbf{y}|A,\theta,\mathcal{H}_1)] = -\frac{A}{2\sigma^2} + \frac{A}{\sigma^2}(y_c\cos\theta + y_s\sin\theta) + \underbrace{c(\mathbf{y})}_{\text{not dependent on A, }\theta}$$
- $$\frac{\partial}{\partial A}\ln[f(\mathbf{y}|A,\theta,\mathcal{H}_1)] = -\frac{A}{\sigma^2} + \frac{y_c\cos\theta + y_s\sin\theta}{\sigma^2} = 0 \Rightarrow$$
$$A = y_c\cos\theta + y_s\sin\theta$$

Example

- $$\frac{\partial}{\partial \theta} \ln[f(\mathbf{y}|A, \theta, \mathcal{H}_1)] = \frac{A}{\sigma^2}(-y_c \sin\theta + y_s \cos\theta) = 0 \Rightarrow$$
$$y_c \sin\theta = y_s \cos\theta \Rightarrow$$
$$\tan\theta = \frac{y_s}{y_c} \Rightarrow$$
$$\hat{\theta}_{ML} = \tan^{-1}\frac{y_s}{y_c} \equiv \phi$$
- set $\theta - \hat{\theta}_{ML} \Rightarrow \hat{A}_{ML} = \frac{y_c^2}{r} + \frac{y_s^2}{r} = \frac{r^2}{r} = r$
- $\hat{\theta}_{ML} = \phi$, $\hat{A}_{ML} = r$
- Thus, $L_G(\mathbf{y}) = \dfrac{f\left(\mathbf{y}|\hat{A}, \hat{\theta}, \mathcal{H}_1\right)}{f\left(\mathbf{y}|\hat{A}, \hat{\theta}, \mathcal{H}_0\right)}$
$$= \frac{\frac{r}{2\pi\sigma^2} e^{-\frac{r^2+r^2}{2\sigma^2}} e^{\frac{r^2}{\sigma^2}\cos(\phi-\phi)}}{\frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}}$$
$$= e^{\frac{r^2}{2\sigma^2}} = e^{\frac{1}{2\sigma^2}\left(y_c^2+y_s^2\right)}$$

Example

- $L_G(\mathbf{y}) \overset{\mathcal{H}_1}{\geq} \tau \Rightarrow$
  $$\frac{1}{2\sigma^2} r^2 \overset{\mathcal{H}_1}{\geq} \ln \tau \Rightarrow$$
  $$r^2 \overset{\mathcal{H}_1}{\geq} 2\sigma^2 \ln \tau \Rightarrow$$
  $$r \overset{\mathcal{H}_1}{\geq} \sigma\sqrt{2 \ln \tau} = \eta$$
- Thus,

$$\Pr(r \geq \eta | \mathcal{H}_0, A = 0) = \int_y^{+\infty} f(r | A = 0, \mathcal{H}_0)\, dr \qquad (2)$$

- From previous results, $f(r, \phi | A = 0, \mathcal{H}_0) = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$
- Thus,

$$f(r | A = 0, \mathcal{H}_0) = \int_0^{2\pi} f(r, \phi | A = 0, \mathcal{H}_0)\, d\phi = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \qquad (3)$$

Example

- From (2) and (3):

$$
\begin{aligned}
\Pr\left(r \geq \eta | \mathcal{H}_0, A = 0\right) &= \int_{\eta}^{+\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr \\
&= \left[-e^{-\frac{r^2}{2\sigma^2}}\right]_{\eta}^{+\infty} \\
&= e^{-\frac{\eta^2}{2\sigma^2}}
\end{aligned}
$$

- $\Pr\left(r \geq \eta | \mathcal{H}_0\right) = a = e^{-\frac{\eta^2}{2\sigma^2}} \Rightarrow$

$$
\begin{aligned}
\ln a &= -\frac{\eta^2}{2\sigma^2} \Rightarrow \\
-2\sigma^2 \ln a &= \eta^2 \Rightarrow \\
-\sigma^2 \ln \frac{1}{a} &= \eta^2 \Rightarrow \\
\eta &= \underbrace{\sigma \sqrt{2 \ln \frac{1}{a}}}_{\text{fully defined}}
\end{aligned}
$$

Example

- Power $\mathrm{P_D} = \int_y^{+\infty} f\left(r|A,\theta,\mathcal{H}_1\right) dr$
-

$$\begin{aligned} f\left(r|A,\theta,\mathcal{H}_1\right) &= \int_0^{2\pi} f\left(r,\phi|A,\theta,\mathcal{H}_1\right) d\phi \\ &= \int_0^{2\pi} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2+A^2}{2\sigma^2}} e^{\frac{Ar}{\sigma^2}\cos(\phi-\theta)} d\phi \\ &= \frac{r}{\sigma^2} e^{-\frac{r^2+A^2}{2\sigma^2}} \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{Ar}{\sigma^2}\cos(\phi-\theta)} d\phi \end{aligned} \tag{4}$$

- Set $I_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{z\cos(\psi)} d\psi$
- Modified Bessel function of zero-th order $I_0(\cdot)$ (monotone increasing function of $z > 0$)
-

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{Ar}{\sigma^2}\cos(\phi-\theta)} d\phi &= \frac{1}{2\pi}\int_{-\theta}^{2\pi-\theta} e^{\frac{Ar}{\sigma^2}\cos(\psi)} d\psi \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{Ar}{\sigma^2}\cos(\psi)} d\psi \end{aligned} \tag{5}$$

Example

- From Eqs. (4) and (5): $f(r|A,\theta,\mathcal{H}_1) = \frac{r}{\sigma^2} e^{-\frac{r^2+A^2}{2\sigma^2}} I_0\left(\frac{Ar}{\sigma^2}\right)$ independent of θ.
- This was expected since $y_c = r\cos\phi$, $y_s = r\sin\phi$, $r = \sqrt{y_c^2 + y_s^2}$. Sufficient statistic r is rotation-invariant - the whole detection problem is rotation-invariant.
- $$\begin{aligned} \mathrm{P_D} &= \Pr(r > \eta|\mathcal{H}_1, A) \\ &\equiv \Pr(r > \eta|A, \mathcal{H}_1) \\ &= \int_{\eta}^{+\infty} \frac{r}{\sigma^2} e^{-\frac{r^2+A^2}{2\sigma^2}} I_0\left(\frac{rA}{\sigma^2}\right) dr \end{aligned}$$
- Marcum's Q Function: $Q_M(a,\beta) \triangleq \int_{\beta}^{+\infty} z e^{-\frac{z^2+a^2}{2}} I_0(az)dz$

  a^2 is called the non-centrality parameter.
- set $z = \frac{r}{\sigma}$
- $\Pr(r > \eta|A) = \int_{\frac{\eta}{\sigma}}^{+\infty} z e^{-\frac{z^2+\left(\frac{A}{\sigma}\right)^2}{2}} I_0\left(z\frac{A}{\sigma}\right) dz = Q_M\left(\frac{A}{\sigma}, \frac{\eta}{\sigma}\right)$.

Asymptotic Optimality of the GLRT I

- For continuous p.d.f. of the form $f(\mathbf{y}|\mathbf{x}) = u(\mathbf{y}) \cdot e^{\left[\mathbf{x}^{\mathrm{T}}\mathbf{s}(\mathbf{y}) - t(\mathbf{x})\right]}$ (exponential family)
- For discrete p.m.f. of the form $\Pr(y_k = i) = p_i$, $i \in \{1, 2, \ldots, k\}$ (multinomial distribution)
  $\mathbf{y} = \left[y_1 \; y_2 \; \ldots \; y_N\right]^{\mathrm{T}}$

GLRT:

1. $\mathrm{P_F}$ (probability of false alarm) has a guaranteed asymptotic exponential decay rate of η:

$$-\lim_{N \to +\infty} \frac{1}{N} \ln \mathrm{P_F}\left(\delta_G,\ N,\ \mathbf{x}_0\right) \geq \eta$$

Asymptotic Optimality of the GLRT II

2. Among all tests that guarantee that the size of the test decays asymptotically at a rate greater or equal to η, the GLRT maximizes the asymptotic rate of decay of the probability of miss $\mathrm{P_M}$:

$$-\lim_{N\to+\infty}\frac{1}{N}\ln \mathrm{P_M}\left(\delta_G,\ N,\ \mathbf{x}_1\right) \geq -\lim_{N\to+\infty}\frac{1}{N}\ln \mathrm{P_M}\left(\delta,\ N,\ \mathbf{x}_1\right)$$

Thank you!

Detection & Estimation Theory: Lectures 22-23

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Kalman Filtering

- Gauss-Markov Process
- Useful Theorem
- Kalman Filter Derivation
- Remarks

Gauss-Markov Process

- *First order Gauss-Markov process:*

$$x[n] = ax[n-1] + u[n], \quad n \geq \emptyset$$

  where $u[n]$ is (zero-mean) white Gaussian noise (WGN) with variance σ_u^2, $x[-1] \sim \mathcal{N}(\mu_s, \sigma_s^2)$, and $x[-1]$ independent of $u[n]$, for all n.
- Are $x[0],\ x[1], \cdots,\ x[n]$ correlated or not? $\rightarrow$ *ANSWER:* of course they are!

$$\begin{aligned} x[0] &= ax[-1] + u[0] \\ x[1] &= ax[0] + u[1] = a(ax[-1] + u[0]) + u[1] \\ &\qquad\qquad\qquad\;\, = a^2x[-1] + au[0] + u[1] \\ &\vdots \\ x[n] &= a^{n+1}x[-1] + \sum_{k=0}^{n} a^k u[n-k] \end{aligned}$$

- $\mathbb{E}\left[x[n]\right] = a^{n+1}\mathbb{E}\left[x[-1]\right] = a^{n+1}\mu_s$ (depends on time, i.e., non-stationary).

Gauss-Markov Process

- x Covariance

$$
\begin{aligned}
\mathbf{C}_s[m,n] &= \mathbb{E}\left[\left(x[m]-\mathbb{E}[x[m]]\right)\left(x[n]-\mathbb{E}[x[n]]\right)\right] \\
&= \mathbb{E}\left[\left\{a^{m+1}(x[-1]-\mu_s)+\sum_{k=0}^{m} a^k u[m-k]\right\}\cdot\right. \\
&\qquad \left.\cdot\left\{a^{n+1}(x[-1]-\mu_s)+\sum_{l=0}^{n} a^l u[n-l]\right\}\right] \\
&\overset{1}{=} a^{n+m+2}\sigma_s^2+\sum_{k=0}^{m}\sum_{l=0}^{n} a^{k+l}\mathbb{E}[u[m-k]u[n-l]] \\
&\overset{2,3}{=} a^{n+m+2}\sigma_s^2+\sum_{k=0}^{m}\sum_{l=0}^{n} a^{k+l}\sigma_u^2\delta[l-(n-m+k)]
\end{aligned}
$$

---

[1] $x[-1]$ independent with $u[n]$

[2] $m-k-(n-l)=m-k-n+l=l-(n-m+k)$

[3] Kronecker $\delta:\delta[u]=1$, when $u=\emptyset$ and $\emptyset$, when $u\neq\emptyset$

Gauss-Markov Process

- We assume $m \geq n$ and $0 \leq l \leq n$, $l = n - m + k$, then

  $$n-m+k \geq 0 \Rightarrow k \geq m-n \quad \text{and} \quad n-m+k \leq n \Rightarrow k \leq m$$

- Then,

$$\begin{aligned}
\mathbf{C}_s[m,n] &= a^{n+m+2}\sigma_s^2 + \sum_{k=m-n}^{m} a^{n-m+2k}\sigma_u^2 \\
&\stackrel{4}{=} a^{n+m+2}\sigma_s^2 + \sigma_u^2 \sum_{k'=0}^{n} a^{2k'+m-n} \\
&= a^{n+m+2}\sigma_s^2 + a^{m-n}\sigma_u^2 \sum_{k=0}^{n} a^{2k}
\end{aligned}$$

- *Properties:*
  - clearly not WSS since depends on time (i.e., n or $n+m$)
  - heavily correlated $|a| \to 1$
  - heavily uncorrelated $|a| \to 0$

[4] $k' = k - (m-n) \Rightarrow 2k = 2k' + 2(m-n)$

Gauss-Markov Process

- for $n > m \Rightarrow \mathbf{C}_s[m,n] = \mathbf{C}_s[n,m]$
- However, Gauss-Markov process for $n \to +\infty$:

$$\mathbb{E}[x[n]] = a^{n+1} \underbrace{\mu_s}_{\mathbb{E}[x[-1]]} \overset{n\to+\infty}{\longrightarrow} \emptyset \quad \text{iff}^5 \quad |a| < 1$$

$$\mathbf{C}_s[m,n] \overset{n\to+\infty}{\longrightarrow} \sigma_u^2 a^{m-n} \sum_{k=0}^{n} a^{2k} = \sigma_u^2 a^{m-n} \frac{1}{1-a^2} \quad \text{for} \quad |a| < 1$$

$$\Rightarrow \mathbf{C}_s[m,n] = \mathbf{C}_s[k = m-n] = \underbrace{\frac{\sigma_u^2}{1-a^2} a^k}_{\text{auto-correlation function}}, \quad k \geq \emptyset \text{ (AR(1) process)}$$

- if $\frac{\sigma_u^2}{1-a^2} = \sigma_s^2$ and $\mu_s = \emptyset$ then the above becomes wide-sense stationary (WSS) for $n \to +\infty$

[5] if and only if

Gauss-Markov Process

- Gauss-Markov Process[6]: mean and variance can be obtained recursively:

$$\mathbb{E}[x[n]] = a\mathbb{E}[x[n-1]] + \mathbb{E}[u[n]] = a\mathbb{E}[x[n-1]]$$

$$\begin{aligned}
\mathrm{var}[x[n]] &= \mathbb{E}\left[(x[n] - \mathbb{E}[x[n]])^2\right] \\
&= \mathbb{E}\left[(ax[n-1] + u[n] - a\mathbb{E}[x[n-1]])^2\right] \\
&= \mathbb{E}\left[\{a(x[n-1] - \mathbb{E}[x[n-1]]) + u[n]\}^2\right] \\
&= a^2\mathrm{var}(x[n-1]) + \sigma_u^2
\end{aligned}$$

since $u[n]$ has zero mean, and $x[n-1]$ depends on $x[-1]$ and $u[0], u[1], \cdots, u[n-1]$ which are independent of $u[n]$!

[6] $x[n] = ax[n-1] + u[n]$

Theorem

- If $\boldsymbol{\theta}$ has zero mean $\mathbb{E}[\boldsymbol{\theta}] = \mathbf{0}$ and $\boldsymbol{\theta}$, $\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}$ are jointly Gaussians, with $\mathbf{y}_1, \mathbf{y}_2$ uncorrelated, then

$$\mathbb{E}[\boldsymbol{\theta}|\mathbf{y}] \stackrel{\triangle}{=} \mathbb{E}[\boldsymbol{\theta}|\mathbf{y}_1, \mathbf{y}_2] = \mathbb{E}[\boldsymbol{\theta}|\mathbf{y}_1] + \mathbb{E}[\boldsymbol{\theta}|\mathbf{y}_2]$$

- *Proof:* since $\boldsymbol{\theta}, \mathbf{y}$ are jointly Gaussians the MSE coincides with the linear least square estimate (LLSE):

$$\hat{\boldsymbol{\theta}} = \mathbb{E}[\boldsymbol{\theta}|\mathbf{y}] = \mathbb{E}[\boldsymbol{\theta}] + \mathbf{C}_{\boldsymbol{\theta}\mathbf{y}} \mathbf{C}_{\mathbf{y}}^{-1} (\mathbf{y} - \mathbb{E}[\mathbf{y}])$$

$$\mathbf{C}_{\mathbf{y}} = \mathbb{E}\left[\begin{bmatrix} \mathbf{y}_1 - \mathbb{E}[\mathbf{y}_1] \\ \mathbf{y}_2 - \mathbb{E}[\mathbf{y}_2] \end{bmatrix} \begin{bmatrix} (\mathbf{y}_1 - \mathbb{E}[\mathbf{y}_1])^{\mathrm{T}} & (\mathbf{y}_2 - \mathbb{E}[\mathbf{y}_2])^{\mathrm{T}} \end{bmatrix} \right]$$

$$\Rightarrow \mathbf{C}_{\mathbf{y}} = \begin{bmatrix} \mathbf{C}_{\mathbf{y}_1} & \mathbf{C}_{12} \\ \mathbf{C}_{12}^{\mathrm{T}} & \mathbf{C}_{\mathbf{y}_2} \end{bmatrix} \tag{1}$$

Theorem

- Notice that $\mathbf{y}_1, \mathbf{y}_2$ are uncorrelated:

$$\mathbf{C}_{12} = \mathbb{E}\left[\begin{bmatrix}(\mathbf{y}_1 - \mathbb{E}[\mathbf{y}_1]) & (\mathbf{y}_2 - \mathbb{E}[\mathbf{y}_2])^{\mathrm{T}}\end{bmatrix}\right]$$
$$= \mathbb{E}\left[\mathbf{y}_1 - \mathbb{E}[\mathbf{y}_1]\right]\mathbb{E}\left[(\mathbf{y}_2 - \mathbb{E}[\mathbf{y}_2])^{\mathrm{T}}\right] = \mathbf{0}\cdot\mathbf{0}^{\mathrm{T}} = \mathbf{0} \qquad (2)$$

- from (1) and (2)

$$\mathbf{C}_{\mathbf{y}} = \begin{bmatrix}\mathbf{C}_{\mathbf{y}_1} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_{\mathbf{y}_2}\end{bmatrix} \Rightarrow \mathbf{C}_{\mathbf{y}}^{-1} = \begin{bmatrix}\mathbf{C}_{\mathbf{y}_1}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_{\mathbf{y}_2}^{-1}\end{bmatrix}$$

$$\mathbf{C}_{\boldsymbol{\theta}\mathbf{y}} \overset{\mathbb{E}[\boldsymbol{\theta}]=\mathbf{0}}{=} \mathbb{E}\left[\boldsymbol{\theta}\cdot\begin{bmatrix}\mathbf{y}_1 - \mathbb{E}[\mathbf{y}_1]\\ \mathbf{y}_2 - \mathbb{E}[\mathbf{y}_2]\end{bmatrix}^{\mathrm{T}}\right] = \begin{bmatrix}\mathbf{C}_{\boldsymbol{\theta}\mathbf{y}_1} & \mathbf{C}_{\boldsymbol{\theta}\mathbf{y}_2}\end{bmatrix}$$

Theorem

- Thus,

$$
\begin{aligned}
\hat{\boldsymbol{\theta}} &= \cancelto{\mathbf{0}}{\mathbb{E}[\boldsymbol{\theta}]} + \begin{bmatrix} \mathbf{C}_{\boldsymbol{\theta}\mathbf{y}_1} & \mathbf{C}_{\boldsymbol{\theta}\mathbf{y}_2} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{\mathbf{y}_1}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\mathbf{y}_2}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 - \mathbb{E}[\mathbf{y}_1] \\ \mathbf{y}_2 - \mathbb{E}[\mathbf{y}_2] \end{bmatrix} \\
&= \mathbf{C}_{\boldsymbol{\theta}\mathbf{y}_1}\mathbf{C}_{\mathbf{y}_1}^{-1}(\mathbf{y}_1 - \mathbb{E}[\mathbf{y}_1]) + \mathbf{C}_{\boldsymbol{\theta}\mathbf{y}_2}C_{\mathbf{y}_2}^{-1}(\mathbf{y}_2 - \mathbb{E}[\mathbf{y}_2]) \\
&= \mathbb{E}[\boldsymbol{\theta}|\mathbf{y}_1] + \mathbb{E}[\boldsymbol{\theta}|\mathbf{y}_2] \qquad \blacksquare
\end{aligned}
$$

Derivation of (scalar) Kalman filter

- Random parameter to be estimated, according to a Gauss-Markov Process:

$$\text{(state equation)} \quad x[n] = ax[n-1] + u[n]$$

  - $x[-1]$ independent of $u[n]$ $\forall n$
  - $u[n]$ WGN (zero mean) with variance σ_u^2

  and

$$\text{(observation equation)} \quad y[n] = x[n] + w[n]$$

  - $u[n]$ zero mean with independent samples and variance σ_u^2 indepedent of time (WGN)
  - $w[n]$ zero mean Gaussian noise with independent samples and $\mathbb{E}\left[w[n]^2\right] = \sigma_n^2$ (depends on time)

- $x[-1] \sim \mathcal{N}(\cancelto{\emptyset}{\mu_s}, \sigma_s^2) \Rightarrow \mathbb{E}\left[x[n]\right] = a^{n+1}\cancelto{\emptyset}{\mu_s} = \emptyset$

Derivation of Kalman filter

- *Problem:* we wish to estimate $x[n]$ base on the observations $\{y[0], y[1], y[2], \cdots, y[n]\}$ or filter $\{y[n]\}$ to produce $\hat{x}[n]$:

$$\hat{x}[n|y[0], y[1], y[2], \cdots, y[m]] \stackrel{\triangle}{=} \hat{x}[n|m]$$

- Optimality criterion: Bayesian MSE: $\mathbb{E}\left[(x[n] - \hat{x}[n|n])^2\right]$

$$\text{MSE:} \quad \hat{x}[n|n] = \mathbb{E}\left[x[n]|y[0], y[1], y[2], \cdots, y[n]\right]$$

$x[n], y[0], y[1], y[2], \cdots, y[n]$ are linear combinations of $x[-1]$ plus Gaussian noise $\Rightarrow x[n], y[0], y[1], y[2], \cdots, y[n]$ are jointly Gaussians!

Derivation of Kalman filter

- Thus MSE $\equiv$ LLSE $\Rightarrow \hat{x}[n|n] = \cancelto{\emptyset}{\mathbb{E}\left[x[n]\right]} + \mathbf{C}_{x\mathbf{y}}\mathbf{C}_{\mathbf{y}}^{-1}\mathbf{y}$ (notice that $\mathbb{E}\left[\mathbf{y}\right] = \mathbf{\emptyset}$ with $\mathbf{y} = \left[y[0], y[1], y[2], \cdots, y[n]\right]^{\mathrm{T}}$)
- If the Gaussian assumption is not valid, then the above holds for the "optimal" LLS estimator. Returning to the original problem:
- We need to find a sequential estimator: if $\{y[n]\}$ were uncorrelated then

$$\begin{aligned}
\hat{x}[n|n] &= \mathbb{E}\left[x[n]|y[0], y[1], y[2], \cdots, y[n]]\right] \\
&= \mathbb{E}\left[x[n]|y[0], y[1], y[2], \cdots, y[n-1]]\right] + \mathbb{E}\left[x[n]|y[n]]\right] \\
&= \hat{x}[n|n-1] + \mathbb{E}\left[x[n]|y[n]]\right]
\end{aligned}$$

however $\{y[n]\}$ are NOT uncorrelated, thus a different approach is needed.

Derivation of Kalman filter

- Set $\tilde{y}[n] = y[n] - \hat{y}[n|n-1]$, where $\hat{y}[n|n-1] = \mathbb{E}\left[y[n]|y[0], \cdots, y[n-1]\right]$ (MSE estimate)
- Thus $\tilde{y}[n] = e$, which $\perp$ to the data $y[0], \cdots, y[n-1]$ (orthogonality principle)
- Set $\mathbf{y}[n-1] = \begin{bmatrix} y[0] & y[1] & \cdots & y[n-1] \end{bmatrix}$
- Thus,

$$y[n] = \tilde{y}[n] + \hat{y}[n|n-1] = \tilde{y}[n] + \sum_{k=0}^{n-1} a_k y[k] \tag{3}$$

since $\{y[k]\}$ jointly Gaussians and thus MSE $\equiv$ LLS (linear).

- $\mathbf{y}[n-1], \tilde{y}[n]$ can give $y[n]$ through (3). Thus $\mathbf{y}[n-1],\ y[n]$ are equivalent to $\mathbf{y}[n-1],\ \tilde{y}[n]$:

$$\hat{x}[n|n] = \mathbb{E}\left[x[n]|\mathbf{y}[n-1], y[n]\right] = \mathbb{E}\left[x[n]|\mathbf{y}[n-1], \tilde{y}[n]\right]$$

Derivation of Kalman filter

- $\tilde{y}$ is error in observation: the trick is to predict $y[n]$, which you already know!
- $\tilde{y}[n]$ uncorrelated with the observation data $\mathbf{y}[n-1]$, due to orthogonality principle.
- Thus,

$$\hat{x}[n|n] = \underbrace{\mathbb{E}\left[x[n]|\mathbf{y}[n-1]\right]}_{\hat{x}[n|n-1]} + \mathbb{E}\left[x[n]|\tilde{y}[n]\right] \tag{4}$$

$$\begin{aligned}\hat{x}[n|n-1] &= \mathbb{E}\left[x[n]|\mathbf{y}[n-1]\right] \\ &= \mathbb{E}\left[ax[n-1]+u[n]|\mathbf{y}[n-1]\right] \\ &= a\mathbb{E}\left[x[n-1]|\mathbf{y}[n-1]\right] \\ &= a\hat{x}[n-1|n-1]\end{aligned} \tag{5}$$

- $u[n]$ is independent of $\{w[n]\}, x[-1]$ and $u[n-1], u[n-2], \cdots, u[0]$

Derivation of Kalman filter

- Thus, $\mathbb{E}\left[u[n]|\mathbf{y}[n-1]\right] = \mathbb{E}\left[u[n]\right] = \emptyset$
- So far:

$$\hat{x}[n|n] = \underbrace{a\hat{x}[n-1|n-1]}_{\hat{x}[n|n-1]} + \mathbb{E}\left[x[n]|\tilde{y}[n]\right] \tag{6}$$

  - $\mathbb{E}\left[x[n]|\tilde{y}[n]\right] = \cancel{\mathbb{E}\left[x[n]\right]}^{\emptyset} + \mathbf{C}_{x\tilde{y}}\mathbf{C}_{\tilde{y}}^{-1}\tilde{y}[n]$
  - $\mathbb{E}\left[\tilde{y}[n]\right] = \emptyset$
  - $\mathbf{C}_{x\tilde{y}} = \mathbb{E}\left[x[n]\tilde{y}[n]\right]$
  - $\mathbf{C}_{\tilde{y}} = \mathbb{E}\left[(\tilde{y}[n])^2\right]$
- Thus,

$$\mathbb{E}\left[x[n]|\tilde{y}[n]\right] = \frac{\mathbb{E}\left[x[n]\tilde{y}[n]\right]}{\mathbb{E}\left[(\tilde{y}[n])^2\right]}\tilde{y}[n] = \underbrace{\frac{\mathbb{E}\left[x[n]\tilde{y}[n]\right]}{\mathbb{E}\left[(\tilde{y}[n])^2\right]}}_{K[n]}\left(y[n] - \hat{y}[n|n-1]\right)$$

$$= K[n]\left(y[n] - \hat{y}[n|n-1]\right)$$

- $\hat{y}[n|n-1] \triangleq \mathbb{E}\left[y[n]|\mathbf{y}[n-1]\right]$

$$\overset{7}{=} \underbrace{\mathbb{E}\left[x[n]|\mathbf{y}[n-1]\right]}_{\hat{x}[n|n-1]} + \cancelto{\emptyset}{\mathbb{E}\left[w[n]|\mathbf{y}[n-1]\right]}$$

- $\mathbb{E}\left[x[n]|\tilde{y}[n]\right] = K[n]\left(y[n] - \hat{y}[n|n-1]\right)$

$$= K[n]\left(y[n] - \hat{x}[n|n-1]\right)$$

- and thus from (6),

$$\hat{x}[n|n] = \underbrace{\hat{x}[n|n-1]}_{a\hat{x}[n-1|n-1]} + K[n]\left(y[n] - \hat{x}[n|n-1]\right)$$

- it remains to calculate the gain $K[n]$ in a recursive manner:

$$K[n] \triangleq \frac{\mathbb{E}\left[x[n]\left(y[n] - \hat{x}[n|n-1]\right)\right]}{\mathbb{E}\left[\left(y[n] - \hat{x}[n|n-1]\right)^2\right]}$$

---

[7] $y[n] = x[n] + w[n]$ and $w[n]$ independent to $y[0], \cdots, y[n-1]$

Derivation of Kalman filter

We observe the following:

- $\mathbb{E}\left[x[n]\left(y[n]-\hat{x}[n|n-1]\right)\right]$

  $= \mathbb{E}\left[\left(x[n]-\hat{x}[n|n-1]\right)\left(y[n]-\hat{x}[n|n-1]\right)\right] \overset{\triangle}{\equiv} M(n|n-1)$
  since

$$\mathbb{E}\left[\underbrace{\hat{x}[n|n-1]}_{\substack{\text{linear combination of}\\ y[0], y[1], \cdots, y[n-1]}} \quad \underbrace{\left(y[n]-\hat{x}[n|n-1]\right)}_{e}\right] \overset{1}{=} \emptyset$$

- and

$$\mathbb{E}\left[w[n]\left(x[n]-\hat{x}[n|n-1]\right)\right] = \emptyset$$

  since $w[n]$ independent (and thus, uncorrelated) to $y[n-1], \cdots, y[0]$.

[1] y[n]=x[n]+w[n]

Derivation of Kalman filter

- Thus,

$$K[n] = \frac{\mathbb{E}\left[\left(x[n]-\hat{x}[n|n-1]\right)\left(x[n]-\hat{x}[n|n-1]\right)\right]}{\mathbb{E}\left[\left(x[n]-\hat{x}[n|n-1]+w[n]\right)^2\right]}$$

$$\overset{8}{\Rightarrow} K[n] = \frac{\mathbb{E}\left[\left(x[n]-\hat{x}[n|n-1]\right)^2\right]}{\mathbb{E}\left[\left(x[n]-\hat{x}[n|n-1]\right)^2\right]+\sigma_n^2} \tag{7}$$

$$\Rightarrow K[n] = \frac{M[n|n-1]}{M[n|n-1]+\sigma_n^2}$$

- ...need recursion:

$$\begin{aligned} M[n|n-1] &\triangleq \mathbb{E}\left[\left(x[n]-\hat{x}[n|n-1]\right)^2\right] \\ &= \mathbb{E}\left[\left(ax[n-1]+u[n]-\hat{x}[n|n-1]\right)^2\right] \\ &= \mathbb{E}\left[\left\{a(x[n-1]-\hat{x}[n-1|n-1])+u[n]\right\}^2\right] \\ &= a^2\mathbb{E}\left[\left(x[n-1]-\hat{x}[n-1|n-1]\right)^2\right]+\sigma_u^2 \end{aligned}$$

[8] numerator: MSE when $\mathbf{y}[n-1]$ is used instead of $\mathbf{y}[n]$

Derivation of Kalman filter

- ...where the last equality is due to the fact that $u[n]$ is independent of $\underbrace{x[0], x[1], \cdots, x[n-1]}_{\text{depend on } x[-1], u[0], \cdots, u[n-1]}$ and

$$\begin{aligned}\mathbf{y}[n-1] &= \begin{bmatrix} y[0] & y[1] & \cdots & y[n-1] \end{bmatrix} \\ &= \begin{bmatrix} x[0]+w[0] & x[1]+w[1] & \cdots & x[n-1]+w[n-1] \end{bmatrix}\end{aligned}$$

$$\Rightarrow \mathbb{E}\left[\left(x[n-1] - \hat{x}[n-1|n-1]\right) u[n]\right] = \emptyset$$

- Thus,

$$\begin{aligned}M[n|n-1] &= a^2 \mathbb{E}\left[\left(x[n-1] - \hat{x}[n-1|n-1]\right)^2\right] + \sigma_u^2 \\ &= a^2 M[n-1|n-1] + \sigma_u^2\end{aligned}$$

Derivation of Kalman filter

- Now we require a recursion for $M[n|n]$:[9]

$$
\begin{aligned}
M[n|n] &= \mathbb{E}\left[(x[n]-\hat{x}[n|n])^2\right] \\
&= \mathbb{E}\left[\{x[n]-\hat{x}[n|n-1] - \underbrace{K[n]}_{\text{constant as a function of n}}(y[n]-\hat{x}[n|n-1])\}^2\right] \\
&= \underbrace{\mathbb{E}\left[(x[n]-\hat{x}[n|n-1])^2\right]}_{M[n|n-1]} \\
&\quad - 2K[n]\underbrace{\mathbb{E}\Big[(x[n]-\hat{x}[n|n-1])(y[n]-\hat{x}[n|n-1])\Big]}_{\text{from (7), numerator of } K[n] \Rightarrow M[n|n-1]} \\
&\quad + K^2[n]\underbrace{\mathbb{E}\left[(y[n]-\hat{x}[n|n-1])^2\right]}_{\text{denominator of } K[n] \Rightarrow \frac{M[n|n-1]}{K[n]}}
\end{aligned}
$$

[9] $\hat{x}[n|n] = \hat{x}[n|n-1] + K[n](y[n] - \hat{x}[n|n-1]$

- Thus,

$$\begin{aligned} M[n|n] &= M[n|n-1] - 2K[n]M[n|n-1] + K[n]M[n|n-1] \\ &= (1 - K[n])\, M[n|n-1]. \end{aligned}$$

[9] $\hat{x}[n|n] = \hat{x}[n|n-1] + K[n](y[n] - \hat{x}[n|n-1]$

Derivation of Kalman filter: Summary

- *Prediction:*
$$\hat{x}[n|n-1] = a\hat{x}[n-1|n-1]$$
- *Prediction MSE:*
$$M[n|n-1] = a^2 M[n-1|n-1] + \sigma_u^2$$
- *Kalman Gain:*
$$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$
- *Correction:*
$$\hat{x}[n|n] = \hat{x}[n|n-1] + K[n](y[n] - \hat{x}[n|n-1]$$
- *Minimum MSE:*
$$M[n|n] = (1 - K[n])M[n|n-1]$$
- ...the same for $\mu_s \neq \emptyset$
- *Initialization:*
$$\hat{x}[-1|-1] = \mathbb{E}[x[-1]] = \mu_s \quad \text{and} \quad M[-1|-1] = \sigma_s^2.$$

Remarks

- *Remark 0:*
  Derived equations hold for $\mu_s \neq \emptyset$ too.
- *Remark 1:*
  - LLS estimates: $\text{LLS}(\boldsymbol{\theta}|\mathbf{y}_1, \mathbf{y}_2) = \text{LLS}(\boldsymbol{\theta}|\mathbf{y}_1) + \text{LLS}(\boldsymbol{\theta}|\mathbf{y}_1)$, where $\mathbf{y}_1,\ \mathbf{y}_2$ uncorrelated
  - orthogonality principle: $\mathbf{e} \perp$ linear combination of the data thus, Kalman filter is the "optimal" (in MSE sense) recursive LINEAR estimator
  - if Gaussian statistics are employed $\Rightarrow$ Kalman is the "optimal" (in MSE sense) estimator!
- *Remark 2:*
  we used Gauss-Markon process for the parameter to be estimated
  - (state equation) $\quad x[n] = ax[n-1] + u[n]$
  - (observation equation) $\quad y[n] = x[n] + w[n]$
  - $\mathbb{E}[(w[n])^2] = \sigma_n^2 \rightarrow$ function of n and $\mathbb{E}[x[n]] = a^{n+1}\mathbb{E}[x[-1]] = a^{n+1}\mu_s$

  thus, Kalman filer holds for non-WSS processes (we haven't seen that so far)!

Remarks

- *Remark 3:*
  Set $a = 1$ and $\sigma_u^2 = \emptyset \Rightarrow x[n] = x[n-1]$ (prediction: last estimate of $x[n]$).
  In that case:

$$\hat{x}[n|n-1] = \hat{x}[n-1]$$
$$M[n|n-1] = M[n-1]$$
$$\hat{x}[n] = \hat{x}[n-1] + K[n](y[n] - \hat{x}[n-1])$$
$$K[n] = \frac{M[n-1]}{M[n-1] + \sigma_n^2}$$
$$M[n] = (1 - K[n])M[n-1]$$

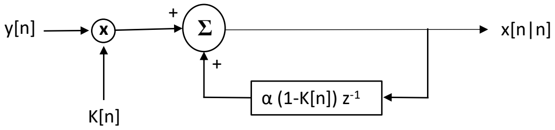
...can omit the prediction stage of Kalman.

Remarks

- *Remark 4:*
  Kalman filter is a time varying linear filter:

$$\hat{x}[n|n] = a\hat{x}[n-1|n-1] + K[n](y[n] - \underbrace{a\hat{x}[n-1|n-1]}_{\hat{x}[n|n-1]})$$

$$= \underbrace{K[n]}_{\text{time varying-constant}} y[n] + \underbrace{a(1-K[n])}_{\text{time varying-constant}} \hat{x}[n-1|n-1]$$



- For $n \to +\infty$ the filter becomes time-invariant (steady-state).

Remarks

- *Remark 5:*
  no-matrix inversion is needed (true only here). For the vector Kalman filter, this is not true.
- *Remark 6:*
  Minimum prediction MSE:

$$\begin{aligned} M[n|n-1] &= \mathbb{E}\left[(x[n]-\hat{x}[n|n-1])^2\right] \\ &= a^2 M[n-1|n-1] + \sigma_u^2 \end{aligned}$$

Kalman Gain:

$$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

Minimum MSE:

$$M[n|n] = (1-K[n])M[n|n-1] \stackrel{\triangle}{=} (1-K[n])\mathbb{E}\left[(x[n]-\hat{x}[n|n-1])^2\right]$$

Thus, $M[n|n]$ can be computed independently of the observation data $\{y[n]\} \Rightarrow$ can be computed offline!

Remarks

- *Remark 7:*
  Kalman filter is a filter: transient response and steady-state response. At steady-state (or $n \to +\infty$) it can be proved that $M[n|n-1] > M[n-1|n-1]$
  Thus, error increases at prediction stage and decreases at correction stage ($K[n] < 1$)

$$M[n|n] = (1 - K[n])M[n|n-1] < M[n|n-1]$$

- *Remark 8:*
  Infinite-length causal Wiener filter:

$$\hat{x}[n] = \sum_{k=0}^{+\infty} h[k]y[n-k]$$

solved through Wiener-Hopf equations.

Remarks

- *Remark 8 continued:*

  (A) We showed that Gauss-Markov $x[n]$ as $n \to +\infty$ could become WSS

  (B) if $\mathbb{E}\left[(w[n])^2\right] = \sigma_n^2 = \sigma^2$ (time independent)

  if (**A**), (**B**) hold then Kalman $\equiv$ Wiener

- *Remark 9:*

  steady-state: time invariant filter for conditions (**A**), (**B**)

$$K[n] = K[\infty]$$

$$\begin{aligned}
\hat{x}[n|n] &= \hat{x}[n|n-1] + K[n](y[n] - \hat{x}[n|n-1]) \\
&= a\hat{x}[n-1|n-1] + K[n](y[n] - a\hat{x}[n-1|n-1]) \\
&= a(1 - K[\infty])\hat{x}[n-1|n-1] + K[\infty]y[n]
\end{aligned}$$

Remarks

- *Remark 9 continued:*

$$\hat{x}[n|n] - a(1-K[\infty])\hat{x}[n-1|n-1] = K[\infty]y[n]$$
$$\Rightarrow \hat{X}(z) - a(1-K[\infty])\hat{X}(z)z^{-1} = K[\infty]Y(z)$$
$$\Rightarrow H(z) = \frac{K[\infty]}{1-a(1-K[\infty])z^{-1}} = H(z=j\omega) = H(z=j2\pi f)$$

- *Remark 10:*
  Same properties for vector Kalman filter (apart from matrix invertibility).

Remarks

- *Remark 10:*
  Equations of the vector Kalman filter can be found in any estimation theory textbook.
- Derivation of the (scalar or vector) Kalman filter equations can be performed in an elegant, simplified way, using (modern) inference theory!
- Kalman filter = Gaussian Belief Propagation in HMMs!
- Pls take the graduate course on ***Probabilistic Graphical Models and Inference Algorithms*** to see this.

Thank you!

Detection & Estimation Theory: Lecture 24

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- Problem Definition & Basic Assumptions
- Prediction/Correction Equations
- Particle Filtering Derivation
  - Importance Sampling
- Remarks

Problem Definition

We limit discussion on scalar case - same reasoning holds for the vector case.

- denote as $p_{n|m} \equiv p_{x_n|y_0,y_1,\cdots,y_m}(x_n|y_0, y_1, \cdots, y_m)$
- again, we follow the same Detection/Estimation notation:
  - y_m is the m-th measurement/observation
  - x_l is the l-th random variable to be estimated ("hidden state")
- General problem:
  estimate $x_0, x_1, \cdots, x_n$ given observations $y_0, y_1, \cdots, y_n$, i.e,

$$\text{find} \quad p_{x_i|y_0,y_1,\cdots,y_n}(x_i|y_0, y_1, \cdots, y_n), \ \underline{0 \leq i \leq n}$$

Prediction/Correction Equations (& Assumptions)

General equations, written in an iterative manner.

- *Prediction Equation:*

$$
\begin{aligned}
&p_{n+1|n}(x_{n+1}|y_0, y_1, \cdots, y_n) = \int_{x_n} p(x_{n+1}, x_n|y_0, y_1, \cdots, y_n)dx_n \\
&= \int_{x_n} p(x_{n+1}|x_n, y_0, y_1, \cdots, y_n)p(x_n|y_0, y_1, \cdots, y_n)dx_n \\
&\overset{1}{=} \int_{x_n} p(x_{n+1}|x_n)p(x_n|y_0, y_1, \cdots, y_n)dx_n \\
&= \int_{x_n} p(x_{n+1}|x_n) \underbrace{p_{n|n}(x_n|y_0, y_1, \cdots, y_n)}_{\text{previous iteration}} dx_n
\end{aligned}
$$

[1] $x_{n+1} \perp y_0, y_1, \cdots, y_n | x_n$

Prediction/Correction Equations (& Assumptions)

- *Update/Correction Equation:*

$$p_{n+1|n+1}(x_{n+1}|y_0, y_1, \cdots, y_{n+1}) = \frac{p(x_{n+1}, y_{n+1}|y_0, y_1, \cdots, y_n)}{p(y_{n+1}|y_0, y_1, \cdots, y_n)}$$

$$= \frac{p(y_{n+1}|x_{n+1}, y_0, y_1, \cdots, y_n)p(x_{n+1}|y_0, y_1, \cdots, y_n)}{\underbrace{p(y_{n+1}|y_0, y_1, \cdots, y_n)}_{z}}$$

$$\stackrel{2}{=} \frac{1}{z} p(y_{n+1}|x_{n+1}) p_{n+1|n}(x_{n+1}|y_0, y_1, \cdots, y_n)$$

- Notice that the above prediction/correction equations hold for:

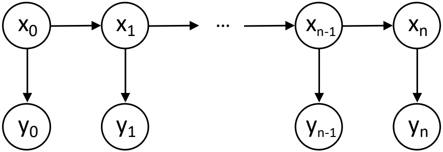
$$x_{n+1} \perp y_0, y_1, \cdots, y_n | x_n \tag{1}$$

$$y_{n+1} \perp y_0, y_1, \cdots, y_n | x_{n+1} \tag{2}$$

[2] $y_{n+1} \perp y_0, y_1, \cdots, y_n | x_{n+1}$

Particle Filter for HMMs with General Continuous

- Eqs. (1), (2) are satisfied by hidden Markov model (HMM):



- HMM satisfies the following:

$$p(x_n|x_{n-1}, x_{n-2}, \cdots, x_0) = p(x_n|x_{n-1}) \tag{3}$$

- Thus, one can produce $(x_0(s), x_1(s), \cdots, x_n(s))$ that adhere to $p_{x_0,x_1,\cdots,x_n}(x_0^n) \triangleq p_{x_0,x_1,\cdots,x_n}(x_0, x_1, \cdots, x_n)$ as follows:

$$x_0(s) \sim p_{x_0}(\cdot),\ \ x_i(s) \sim p_{x_i|x_{i-1}}(\cdot|x_{i-1}(s)),\ \ i = 1, \cdots, n \tag{4}$$

- Note that due to (3),

$$p_{x_0,x_1,\cdots,x_n}(x_0^n) = p_{x_0}(x_0) \prod_{i=1}^{n} p_{x_i|x_{i-1}}(x_i|x_{i-1})$$

Particle Filter Derivation

- HMM also satisfies the following:

$$p(y_n|x_n, x_{n-1}, y_{n-1}, x_{n-2}, y_{n-2}, \cdots, x_0, y_0) = p(y_n|x_n)$$

thus,

$$\begin{aligned}
p(y_n, y_{n-1}, \cdots, y_0|x_n, x_{n-1}, \cdots, x_0) = \\
= p(y_n|y_{n-1}, \cdots, y_0, x_n, x_{n-1}, \cdots, x_0) \\
\cdot\, p(y_{n-1}, \cdots, y_0|x_n, x_{n-1}, \cdots, x_0) \\
\overset{(6)}{=} p(y_n|x_n)p(y_{n-1}, \cdots, y_0|x_n, x_{n-1}, \cdots, x_0)
\end{aligned}$$

Working inductively,

$$p(y_n, y_{n-1}, \cdots, y_0|x_n, x_{n-1}, \cdots, x_0) = \prod_{i=0}^{n} p_{y_i|x_i}(y_i|x_i) \quad (5)$$

Particle Filter Derivation

- x_{n-1} separates $y_{n-1}, \cdots, y_0$ from x_n and thus (6):

$$y_{n-1}, \cdots, y_0 \perp x_n | x_{n-1}, x_{n-2}, \cdots, x_0 \tag{6}$$

- In HMM-particle filtering, we are given $p(x_i|x_{i-1})$ and $p(y_i|x_i),\ 0 \leq i \leq n$
- Thus,[3]

$$p_{x_0,x_1,\cdots,x_n|y_0,y_1,\cdots,y_n}(x_0^n|y_0^n) = \frac{p(y_0^n|x_0^n)p(x_0^n)}{p(y_0^n)}$$

$$= \frac{p_{x_0,x_1,\cdots,x_n}(x_0^n)\prod_{i=0}^{n} p_{y_i|x_i}(y_i|x_i)}{\underbrace{p_{y_0,y_1,\cdots,y_n}(y_0^n)}_{z \,\rightarrow\, \text{unknown (i.e., hard to compute) constant}}} \tag{7}$$

[3] notation: $x_0^n = x_0, x_1, \cdots, x_n$ and $y_0^n = y_0, y_1, \cdots, y_n$

Particle Filter Idea

- Particle filtering idea:
  1) Produce samples that adhere to $p_{x_0,x_1,\cdots,x_n|y_0,y_1,\cdots,y_n}(x_0^n|y_0^n)$ without knowing z.
  2) These samples can be used to estimate any $\underset{p_{x_0^n|y_0^n}}{\mathbb{E}}[f(x_0^n)]$, for any function $f(\cdot)$ as if one perfectly knew the p.d.f. $p_{x_0^n|y_0^n}(\cdot|y_0^n)$.
- We do not know z; can only produce as many samples $\{(x_0(s), x_1(s), \cdots, x_n(s)\},\ s = 1, \cdots, \mathcal{S}$ according to known $\prod_{i=0}^{n} p_{y_i|x_i}(y_i|x_i)$
- Particle filtering is a a case of non-parametric modeling, since we do not know the posterior p.d.f. in closed form $p_{x_0^n|y_0^n}(\cdot|y_0^n)$.

Example

- In localization research, we need the conditional mean (i.e., MMSE estimator):

$$\mathop{\mathbb{E}}_{p_{x_0^n | y_0^n}} \left[f(x_0^n) \right] \equiv \mathbb{E} \left[x_n | y_0, y_1, \cdots, y_n \right]$$

Importance Sampling

- Theory to solve the above problem $\Rightarrow$ *Importance Sampling:*

  $$\mu(x) = \frac{q(x)}{z} \begin{array}{l} \leftarrow \text{ known function} \\ \leftarrow \text{ unknown function} \end{array}$$

  We want samples from $\mu(\cdot)$ so that we can compute the following:

  $$\mathbb{E}_{\mu}\left[f(x)\right] \triangleq \int f(x)\mu(x)dx$$

Important Sampling Algorithm

- *Important Sampling algorithm:*
  1) Produce samples $x(1), x(2), \cdots, x(s), \cdots, x(\mathcal{S})$ from known distribution $v(x)$, called the "proposal" distribution;
  2) Compute as many weights as the samples, according to

$$w(s) = \frac{q(x(s))}{v(x(s))}, \quad s = 1, 2, \cdots, \mathcal{S}$$

  Calculate $\hat{E}(\mathcal{S})$ instead of $\mathbb{E}_\mu [f(x)]$:

$$\mathbb{E}_\mu [f(x)] \to \hat{E}(\mathcal{S}) = \frac{\frac{1}{\mathcal{S}} \sum_{s=1}^{\mathcal{S}} w(s) f(x(s))}{\frac{1}{\mathcal{S}} \sum_{s=1}^{\mathcal{S}} w(s)}$$

- *Definition:* support of function $p(x)$(supp(p))

$$\text{supp}(p) = \{x : p(x) > 0\}$$

Important Sampling Algorithm: Theorem

- *Theorem 1:* Let $\text{supp}(\mu) \subseteq \text{supp}(v)$. Then for $\mathcal{S} \to +\infty$

$$\hat{E}(\mathcal{S}) \to \mathbb{E}_{\mu}\left[f(x)\right], \quad \text{with probability 1}$$

- *Proof:*

$$\lim_{\mathcal{S}\to+\infty} \frac{1}{\mathcal{S}} \sum_{s=1}^{\mathcal{S}} w(s) f(x(s)) \xrightarrow[\text{Large numbers}]{\text{strong Law of}} \underbrace{\mathbb{E}_v\left[w \cdot f(x)\right]}_{\substack{\text{expected value in terms of} \\ v(x) \text{ because we have samples from} \\ \text{the "proposal" pdf } v(x)}}$$

$$\mathbb{E}_v\left[w \cdot f(x)\right] = \mathbb{E}_v\left[\frac{q(x)}{v(x)} f(x)\right] = \int_{\text{supp}(v)} \frac{q(x)}{\cancel{v(x)}} f(x) \cancel{v(x)} dx =$$

$$= \int_{\text{supp}(v)} q(x) f(x) dx \stackrel{4}{=} z \int_{\text{supp}(\mu)} \mu(x) f(x) dx = z\mathbb{E}_{\mu}[f(x)] \quad (8)$$

[4] $q(x) = 0 \ \forall \ x \notin \text{supp}(\mu)$

Important Sampling Algorithm: Theorem

- *Proof continued:*
  similarly,

$$\lim_{\mathcal{S}\to+\infty}\frac{1}{\mathcal{S}}\sum_{s=1}^{\mathcal{S}} w(s) \xrightarrow[\text{Large numbers}]{\text{strong Law of}} \mathbb{E}_v[w] = \int_{\mathrm{supp}(v)} \frac{q(x)}{\cancel{v(x)}}\cancel{v(x)}dx \overset{5}{=}$$

$$= \int_{\mathrm{supp}(\mu)} q(x)dx =$$

$$= z\cancelto{1}{\int_{\mathrm{supp}(\mu)} \mu(x)dx} = z \tag{9}$$

from (8) and (9) $\Rightarrow$ proof is completed. ■

[5] $q(x) = 0 \;\forall\; x \notin \mathrm{supp}(\mu)$

Remarks

- It is worth noting that as long as $\mathrm{supp}(\mu)$ is contained in $\mathrm{supp}(v)$, the estimation converges irrespective of the choice of v.

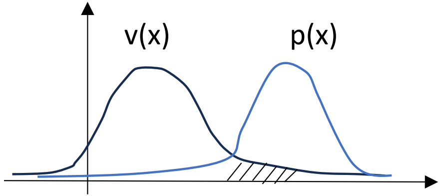


Figure 1: Area where proposal distribution obtains small values, as opposed to true distribution; that amplifies the number of required samples.

- However, the choice of v determines the variance of the estimator and hence the number of samples $\mathcal{S}$ required to obtain good estimation.

Back to HMM-PF

- Need to estimate $\underset{p_{x_0^n|y_0^n}}{\mathbb{E}}[f(x_0^n)|y_0^n]$
- Set

$$q(x_0^n) = p_{x_0,x_1,\cdots,x_n}(x_0^n)\prod_{i=0}^{n} p_{y_i|x_i}(y_i|x_i)$$

- Set

$$v(x_0^n) = \text{ prior } = p_{x_0,x_1,\cdots,x_n}(x_0^n)$$

- For any given $(x_0, x_1, \cdots, x_n) \equiv x_0^n$ sample, calculate the corresponding weight as follows:

$$w(x_0^n) \equiv w_0^n = \frac{q(x_0^n)}{v(x_0^n)} \overset{(7)}{=} \prod_{i=0}^{n} p_{y_i|x_i}(y_i|x_i)$$

HMM-PF Summary

- Thus, for given $y_0, y_1, \cdots, y_n = y_0^n$
  1) Obtain $\mathcal{S}$ iid samples $x_0^n(s),\ 1 \leq s \leq \mathcal{S}$ according to $p_{x_0, x_1, \cdots, x_n}(\cdot)$
  2) Compute $\mathcal{S}$ corresponding weights
  $$w_0^n(s) = \prod_{i=0}^{n} p(y_i | x_i(s)) \quad \forall s,\ 1 \leq s \leq \mathcal{S}$$
  3) Output estimation of $\underset{p_{x_0^n | y_0^n}}{\mathbb{E}} [f(x_0, x_1, \cdots, x_n) | y_0, y_1, \cdots, y_n]$

$$\text{as} \quad \frac{\sum_{s=1}^{\mathcal{S}} w_0^n(s) f(x_0^n(s))}{\sum_{s=1}^{\mathcal{S}} w_0^n(s)}.$$

Remark 1

- The above estimator can be implemented in a sequential fashion, exploiting the HMM properties:
  - for given s in $\{1, 2, \cdots, \mathcal{S}\}$ sample $x_{k+1}(s) \sim p_{x_{k+1}|x_k}(\cdot|x_k(s))$
  - compute

$$\underbrace{w_0^{k+1}(s)}_{\substack{\text{corresponding}\\\text{weight for}\\\text{particle } x_{k+1}(s)}} = \underbrace{w_0^{k+1}(s)}_{\substack{\text{weight for}\\\text{sample } x_k(s)}} \; p_{y_{k+1}|x_{k+1}}(y_{k+1}|x_{k+1}(s))$$

  - repeat for all k up to n and all s up to $\mathcal{S}$
  - re-sample the weights/particles before you proceed to next $k+1$ (particle depletion problem)

Remark 2

- Can we extend particle filtering to other probabilistic graphical models (PGM), other than HMMs?
- *Answer:* YES, using mixture of Gaussians and producing samples based on specific Markov chain Monte Carlo (MCMC) techniques: say $p(x) = \frac{q(x)}{z}$ ($z \rightarrow$ unknown constant)
  You can craft a Markov chain (MC) that produces samples according to $p(x)$, even though the MC was built using $q(x) \rightarrow$ Metropolis-Hastings technique (Gibbs sampling is a special case).
- ...you can take graduate class ***Probabilistic Graphical Models and Inference Algorithms*** to see the above!

Thank you!